602224

C

F-TS-9811/V

TT-60-23871-2

THE THEORY OF RANDOM PROCESSES AND ITS APPLICATION IN RADIO ENGINEERING

225 pg

BY: B. R. Levin

(PART II of II, pages 254 - 475)

602224

CODE-23

F-TS-9811/V

CLEARINGHOUSE FOR FEDERAL SCIENTIFIC AND TECHNICAL INFORMATION CFSTI
DOCUMENT MANAGEMENT BRANCH 410.11

LIMITATIONS IN REPRODUCTION QUALITY

ACCESSION # AD 602 224

- [x] 1. WE REGRET THAT LEGIBILITY OF THIS DOCUMENT IS IN PART UNSATISFACTORY. REPRODUCTION HAS BEEN MADE FROM BEST AVAILABLE COPY.
- [x] 2. A PORTION OF THE ORIGINAL DOCUMENT CONTAINS FINE DETAIL WHICH MAY MAKE READING OF PHOTOCOPY DIFFICULT.
- [ ] 3. THE ORIGINAL DOCUMENT CONTAINS COLOR, BUT DISTRIBUTION COPIES ARE AVAILABLE IN BLACK-AND-WHITE REPRODUCTION ONLY.
- [ ] 4. THE INITIAL DISTRIBUTION COPIES CONTAIN COLOR WHICH WILL BE SHOWN IN BLACK-AND-WHITE WHEN IT IS NECESSARY TO REPRINT.
- [ ] 5. LIMITED SUPPLY ON HAND: WHEN EXHAUSTED, DOCUMENT WILL BE AVAILABLE IN MICROFICHE ONLY.
- [ ] 6. LIMITED SUPPLY ON HAND: WHEN EXHAUSTED DOCUMENT WILL NOT BE AVAILABLE.
- [ ] 7. DOCUMENT IS AVAILABLE IN MICROFICHE ONLY.
- [ ] 8. DOCUMENT AVAILABLE ON LOAN FROM CFSTI (TT DOCUMENTS ONLY).
- [ ] 9.

NBS 9/64

PROCESSOR:

TEORIYA SLUCHAYNYKH PROTSESSOV I EE PRIMENENIYE B RADIOTECHNIKE

Izdatel'stvo "Sovetskoye Radio"

Moscow 1957

Foreign Pages: 496

F-TS-9811/V

Chapter VII

NONLINEAR TRANSFORMATIONS OF THE NORMAL RANDOM PROCESS

(Power spectra)

1. General Solution, Obtained by Correlation Method.

The normal random process occupies a central place in the majority of the practical applications of the theory being discussed here. Therefore a systematic exposition is in order, of the transformations undergone by a normal random process in its passage through radio-equipment components of various types.

It has been noted above, that the problem of the passage of a normal random process through linear systems is a comparatively simple one, since the process retains its normal distribution at the output; only the correlation function of the process, and the power spectrum corresponding to it, are subject to change. All formulas necessary to the computations are contained in Section 2, Ch.VI. Therefore, the principal interest is presented by the problem of the nonlinear transformation of a normal random process.

In the present chapter the indicated problem will be restricted to a study only of the correlation function and power spectrum of a process at the output of a nonlinear system. For this we shall employ the general methods indicated in Section 6, Ch. VI.

Let, at the input of a nonlinear system, there act a random process constituting a sum of the determined process (e.g., signal) S(t) and a stationary normal random process with a zero mean value (e.g., fluctuation noises). In accordance with (5.96), the two-dimensional distribution function of this process has the form of

$$w_2(x_1, x_2, t, \tau) = \frac{1}{2\pi\sigma^2\sqrt{1-R^2}} e^{-\frac{(x_1-a_1)^2 + (x_2-a_2)^2 - 2R(x_1-a_1)(x_2-a_2)}{2\sigma^2(1-R^2)}}, \qquad (7.1)$$

where $a_1 = S(t)$, $a_2 = S(t+\tau)$, while $R = R(\tau)$ and σ^2 is the correlation coefficient and dispersion of the stationary portion of the random process.

Substituting (7.1) into (6.5), we obtain the expression for the correlation function of the random process at the output of a nonlinear system with a characteristic of $y = f(x)$, at the input of which there acts a determined signal in sum with a stationary normal random process

$$B(\tau, t) = \frac{1}{2\pi\sigma^2\sqrt{1-R^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1) f(x_2) \times \\ \times e^{-\frac{(x_1-a_1)^2 - 2R(x_1-a_1)(x_2-a_2) + (x_2-a_2)^2}{2\sigma^2(1-R^2)}} dx_1 dx_2. \tag{7.2}$$

Effecting in (7.2) the substitution of x_1 for σx_1 and of x_2 for σx_2, and adopting the designations

$$\alpha_1 = \frac{S(t)}{\sigma}, \quad \alpha_2 = \frac{S(t+\tau)}{\sigma},$$

we reduce the expression for the correlation function to the form of

$$B(\tau, t) = \frac{1}{2\pi\sqrt{1-R^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\sigma x_1) f(\sigma x_2) \times \\ \times e^{-\frac{(x_1-\alpha_1)^2 - 2R(x_1-\alpha_1)(x_2-\alpha_2) + (x_2-\alpha_2)^2}{2(1-R^2)}} dx_1 dx_2. \tag{7.3}$$

For computation of the double integral in (7.3) we shall employ the first of the methods indicated in Section 6, Ch. VI.

In the case under discussion, the one-dimensional distribution functions corresponding to $w_2(\sigma x_1, \sigma x_2, \tau)$ are equal to

$$w_{11}(x_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_1-\alpha_1)^2}{2}}, \\ w_{12}(x_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_2-\alpha_2)^2}{2}}, \tag{7.4}$$

with the variables x_1 and x_2 covering the range of from $-\infty$ to $+\infty$. If the functions (7.4) are adopted as weighting functions, then, as is known (cf. the book of V. L. Gonchareov cited on p. 241), the aggregates of orthogonal polynomials corresponding

to them are the Hermite polynomials $H_n(x_1 - \alpha_1)$ and $H_n(x_2 - \alpha_2)$ (cf. Appendix VII). An expansion of (6.57) for a two-dimensional normal distribution function has the following form:

$$\frac{1}{2\pi\sqrt{1-R^2}} e^{-\frac{(x_1-a_1)^2 - 2R(x_1-a_1)(x_2-a_2) + (x_2-a_2)^2}{2(1-R^2)}} =$$

$$= \frac{1}{2\pi} e^{-\frac{(x_1-a_1)^2 + (x_2-a_2)^2}{2}} \sum_{n=0}^{\infty} \frac{R^n}{n!} H_n(x_1-\alpha_1) H_n(x_2-\alpha_2)^{*}. \qquad (7.5)$$

We substitute the series thus obtained under the integral sign in expression (7.3)

$$B(\tau,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\sigma x_1) f(\sigma x_2) \sum_{n=0}^{\infty} \frac{R^n}{n!} H_n(x_1-\alpha_1) H_n(x_2-\alpha_2) \times$$
$$\times e^{-\frac{(x_1-a_1)^2 + (x_2-a_2)^2}{2}} dx_1 dx_2.$$

Changing the order of summation and integration, and noting that the variables of integration are thereby separated, we find

$$B(\tau,t) = \sum_{n=0}^{\infty} \frac{R^n}{n!} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\sigma x_1) H_n(x_1-\alpha_1) e^{-\frac{(x_1-\alpha_1)^2}{2}} dx_1 \right) \times$$
$$\times \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\sigma x_2) H_n(x_2-\alpha_2) e^{-\frac{(x_2-a_2)^2}{2}} dx_2 \right). \qquad (7.6)$$

We introduce the designation

$$c_n = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\sigma x) H_n(x-\alpha) e^{-\frac{(x-a)^2}{2}} dx. \qquad (7.7)$$

Then from (7.6) we obtain the desired expression for the correlation function of a random process at the output of a nonlinear system

$$B(\tau,t) = \sum_{n=0}^{\infty} c_{1n} c_{2n} \frac{R^n}{n!}, \qquad (7.8)$$

where c_{1n} and c_{2n} are obtained from c_n, if in (7.7) in place of α are respectively

* Expansion (7.5) may be obtained without difficulty if, with the employment of integral representation of a two-dimensional normal distribution function in terms of a two-dimensional characteristic function and, with the expansion into a series of the exponential multiplier containing a product of the variables in the integrand expression, the integrals are expressed in terms of Hermite polynomials.

assumed α_1 and α_2. Since in the general case correlation function (7.8) depends on time, therefore, before calculating the power spectrum, it is necessary to average this correlation function over time

$$B^{*}(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{-\frac{T}{2}}B(\tau,t)\,dt. \tag{7.9}$$

Since in the series (7.8) only the coefficients c_{1n} and c_{2n} are functions of time,

$$B^{*}(\tau)=\sum_{n=0}^{\infty}c_n^{*}\frac{R^n}{n!}, \tag{7.10}$$

where

$$c_n^{*}(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{-\frac{T}{2}}c_{1n}c_{2n}\,dt. \tag{7.11}$$

Subjecting (7.10) to a Fourier transformation, we obtain an explicit expression for the power spectrum of a normal random process which has passed through a nonlinear system

$$F(\omega)=4\int_0^{\infty}B^{*}(\tau)\cos\omega\tau\,d\tau=4\sum_{n=0}^{\infty}\frac{1}{n!}\int_0^{\infty}c_n^{*}(\tau)R^n(\tau)\cos\omega\tau\,d\tau. \tag{7.12}$$

2. Case of Stationary and Narrow-band Input Process.

Let the determined portion of a normal process be lacking. Then $\alpha_1 = \alpha_2 = 0$, and from (7.8) we obtain the expression for the correlation function of the process at the output of a nonlinear system upon the input of which there acts a stationary normal random process (e.g., noise in the absence of a signal)

$$B(\tau)=\sum_{n=0}^{\infty}c_n^{2}\frac{R^n(\tau)}{n!}, \tag{7.13}$$

where

$$c_n=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(\sigma x)H_n(x)e^{-\frac{x^2}{2}}dx. \tag{7.14}$$

The first term in the series (7.13) corresponds to the direct component (discrete portion of the spectrum), and the sum of the remaining terms, to the continuous portion of the power spectrum of a random process at the output of a linear system.

Let the power spectrum of a stationary normal process be a narrow-band one, i.e., let it be concentrated in a relatively narrow frequency band about the high frequency ω_0, at which spectral density is at its maximum and with respect to which the spectrum may be considered symmetrical. An example of such a process may be found in noise at the output of a linear system, the band of which is much smaller than its resonance frequency. In accordance with (5.82) the correlation coefficient may be represented in the form of

$$R(\tau)=R_0(\tau)\cos\omega_0\tau, \tag{7.15}$$

where $R_0(\tau)$ is the correlation coefficient of the envelope of the random process at hand. Substituting (7.15) into (7.13), we find

$$B(\tau)=\sum_{n=0}^{\infty} c_n^2 \frac{R_0^n(\tau)}{n!}\cos^n\omega_0\tau. \tag{7.16}$$

We replace the powers of the cosines in (7.16) by the sum of the cosines of the multiple arcs according to the well-known formulas

$$\cos^{2n}x=\frac{1}{2^{2n}}\left[\sum_{k=0}^{n-1} 2\binom{2n}{k}\cos 2(n-k)x+\binom{2n}{n}\right], \tag{7.17}$$

$$\cos^{2n-1}x=\frac{1}{2^{2n-2}}\sum_{k=0}^{n-1}\binom{2n-1}{k}\cos(2n-2k-1)x. \tag{7.18}$$

Then expression (7.16) of the correlation function of the process at the output of a nonlinear system will take the form of

$$B(\tau)=\sum_{n=0}^{\infty} c_{2n}^2 \frac{\binom{2n}{n}}{(2n)!\,2^{2n}} R_0^{2n}(\tau)+$$

$$+\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\frac{c_{2n-1}^{2}}{(2n-1)!}\frac{\binom{2n-1}{k}}{2^{2n-2}}R_0^{2n-1}(\tau)\cos(2n-2k-1)\,\omega_0\tau+$$

$$+\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\frac{c_{2n}^{2}}{(2n)!}\frac{\binom{2n}{k}}{2^{2n-1}}R_0^{2n}(\tau)\cos 2(n-k)\,\omega_0\tau. \tag{7.19}$$

Adopting the designation $r = n - k$ and changing the order of summation in the double sums, we obtain

$$\begin{aligned} B(\tau)=&\sum_{n=0}^{\infty}c_{2n}^{2}\frac{\binom{2n}{n}}{(2n)!\,2^{2n}}R_0^{2n}(\tau)+\\ &+\left[\sum_{n=1}^{\infty}\frac{c_{2n-1}^{2}}{(2n-1)!}\frac{\binom{2n-1}{n-1}}{2^{2n-2}}R_0^{2n-1}(\tau)\right]\cos\omega_0\tau+\\ &+\sum_{r=2}^{\infty}\left[\sum_{n=r}^{\infty}\frac{c_{2n-1}^{2}}{(2n-1)!}R_0^{2n-1}(\tau)\right]\cos(2r-1)\,\omega_0\tau+\\ &+\sum_{r=1}^{\infty}\left[\sum_{n=r}^{\infty}\frac{c_{2n}^{2}}{(2n)!}\frac{\binom{2n}{n-r}}{2^{2n-1}}R_0^{2n}(\tau)\right]\cos 2r\omega_0\tau. \end{aligned} \tag{7.20}$$

Designating

$$B_0(\tau)=\sum_{n=0}^{\infty}c_{2n}^{2}\frac{\binom{2n}{n}}{(2n)!\,2^{2n}}R_0^{2n}(\tau), \tag{7.21}$$

$$B_{2r-1}(\tau)=\sum_{n=r}^{\infty}\frac{c_{2n-1}^{2}}{(2n-1)!}\frac{\binom{2n-1}{n-r}}{2^{2n-2}}R_0^{2n-1}(\tau), \tag{7.22}$$

$$B_{2r}(\tau)=\sum_{n=r}^{\infty}\frac{c_{2n}^{2}}{(2n)!}\frac{\binom{2n}{n-r}}{2^{2n-1}}R_0^{2n}(\tau), \tag{7.23}$$

we rewrite (7.20) in the form of

$$\begin{aligned} B(\tau)=&B_0(\tau)+B_1(\tau)\cos\omega_0\tau+\\ &+\sum_{r=2}^{\infty}B_{2r-1}(\tau)\cos(2r-1)\,\omega_0\tau+\sum_{r=1}^{\infty}B_{2r}(\tau)\cos 2r\omega_0\tau. \end{aligned} \tag{7.24}$$

The power spectrum is, in accordance with Khinchin's theorem, equal to a Fourier transformation of $B(\tau)$. If the Fourier transformation of each of the items in (7.24) is designated by $F_r(\omega)$, i.e.,

$$F_r(\omega)=4\int_0^{\infty}B_r(\tau)\cos r\omega_0\tau\cos\omega\tau\,d\tau, \tag{7.25}$$

then we obtain the following expression for the power spectrum of the random process at the output of a nonlinear system

$$F(\omega)=F_0(\omega)+F_1(\omega)+\sum_{r=2}^{\infty}F_r(\omega). \tag{7.26}$$

A graphical representation of this spectrum is shown in Figure 48 (the dotted line indicates the spectrum of the normal, stationary process acting on the input of the system).

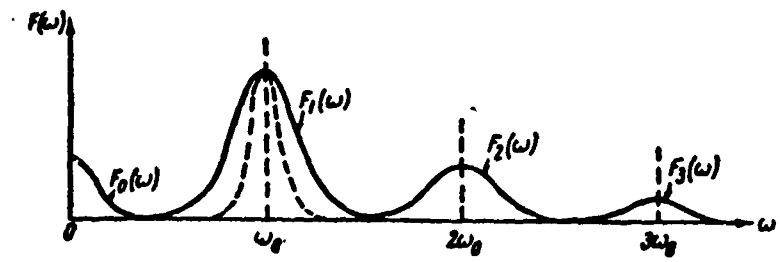


Fig. 48. Power spectrum of narrow-band random process after nonlinear transformation.

The first term in (7.26) represents the low-frequency portion of the power spectrum (the so-called video spectrum) of the random process at the output of a nonlinear system. The second term corresponds to the portion of the power spectrum of the output process lying about frequency ω_0, where is also concentrated the spectrum of the input process. The remaining terms in (7.26) correspond to the high-frequency portions of the power spectrum of the process at the output of a nonlinear system, which lie about the odd and even harmonics of frequency ω_0.

The video spectrum $F_0(\omega)$ is of greatest interest in the study of demodulation processes in radio receivers, whereas the spectrum band $F_1(\omega)$ is important to the study of the modulation process in radio transmitters.

From (7.21) - (7.23) it can be seen that for the computation of a power spectrum it is necessary to obtain inverse Fourier transformations of the powers of the correlation coefficients $R_0^k(\tau)$. The higher is k, the less are the spectrum densities corresponding to $R_0^k(\tau)$, but the wider is the frequency band occupied by the spectrum. For large instances of k, computation of the power-spectrum component which cor-

responds to $R_0^k(\tau)$ is complex. However function $R_0^k(\tau)$ diminishes so rapidly, that an appropriate approximation may be employed. Thus, for instance, if the spectrum of a random process is uniform in band Δ, then according to (6.18)

$$R_0(\tau)=\frac{\sin\frac{\tau\Delta}{2}}{\frac{\tau\Delta}{2}}\approx 1-\frac{1}{6}\left(\frac{\tau\Delta}{2}\right)^2.$$

Since $e^{-\beta\left(\frac{\tau\Delta}{2}\right)^2}\approx 1-\beta\left(\frac{\tau\Delta}{2}\right)^2+\dots$, there is permissible the approximation

$$R_0^k(\tau)=\left(\frac{\sin\frac{\tau\Delta}{2}}{\frac{\tau\Delta}{2}}\right)^k\approx e^{-\frac{k}{6}\left(\frac{\tau\Delta}{2}\right)^2},$$

and the power spectrum (inverse Fourier transformation) corresponding to this approximation is equal to

$$2\int_{-\infty}^{\infty}R_0^k(\tau)e^{-i\omega\tau}d\tau\approx\frac{4}{\Delta}\sqrt{\frac{6\pi}{k}}\,e^{-\frac{3}{2k}\left(\frac{2\omega}{\Delta}\right)^2}.$$

Figure 49 shows the power spectra corresponding to $\left(\frac{\sin\frac{\tau\Delta}{2}}{\frac{\tau\Delta}{2}}\right)^k$ for k = 1, 2,...6.

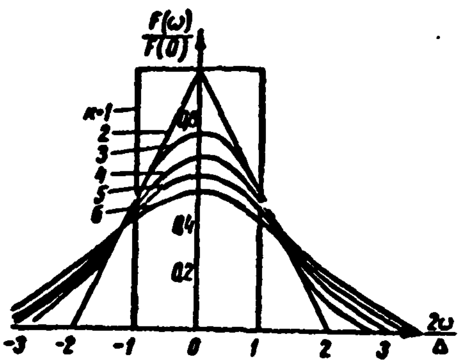


Fig. 49. Spectra of function $\left(\frac{\sin\frac{\tau\Delta}{2}}{\frac{\tau\Delta}{2}}\right)^k$.

3. Linear Detector.

As the first example illustrating the method set forth above, let us consider the manner in which the correlation function and spectrum of a normal random process are transformed in its passage through a linear detector*, whose characteristic has

* Here and in the future, in accordance with Section 1 Ch.VI detection is regarded only as a nonlinear and noninertial process. The subsequent action of the filtering element must be treated separately.

the form of

$$y=f(x)=\begin{cases} x & x>0, \\ 0 & x<0 \end{cases} \tag{7.27}$$

(the constant multiplier for x is assumed equal to unity, which is not essential since it serves the purpose of scale and may always be taken into account in the final results).

The coefficients c_n in series (7.13) are in the case at hand obtained from the integral

$$c_n=\frac{\sigma}{\sqrt{2\pi}}\int_0^\infty xH_n(x)e^{-\frac{x^2}{2}}dx.$$

When n = 0 and n = 1 we obtain directly

$$c_0=\frac{\sigma}{\sqrt{2\pi}}\int_0^\infty xe^{-\frac{x^2}{2}}dx=\frac{\sigma}{\sqrt{2\pi}}. \tag{7.28}$$

$$c_1=\frac{\sigma}{\sqrt{2\pi}}\int_0^\infty x^2e^{-\frac{x^2}{2}}dx=\frac{\sigma}{2}. \tag{7.29}$$

When $n\geqslant 2$, we obtain by integrating by parts

$$c_n=\frac{(-1)^n\sigma}{\sqrt{2\pi}}\int_0^\infty x\frac{d^n e^{-\frac{x^2}{2}}}{dx^n}dx=\frac{(-1)^{n+1}\sigma}{\sqrt{2\pi}}\int_0^\infty \frac{d^{n-1}e^{-\frac{x^2}{2}}}{dx^{n-1}}dx=$$

$$=\frac{(-1)^{n+1}\sigma}{\sqrt{2\pi}}\frac{d^{n-2}}{dx^{n-2}}\left(e^{-\frac{x^2}{2}}\right)\Big|_0^\infty.$$

or

$$c_n=\frac{\sigma}{\sqrt{2\pi}}H_{n-2}(0). \tag{7.30}$$

Substituting (7.28) - (7.30) into (7.13), we find

$$B(\tau)=\frac{\sigma^2}{2\pi}\left[1+\frac{\pi}{2}R(\tau)+\sum_{n=2}^\infty H_{n-2}^2(0)\frac{R^n(\tau)}{n!}\right]$$

Bearing in mind (cf. Appendix VII), that

$$H_{2k}(0)=(-1)^k(2k-1)!!,\quad H_{2k-1}(0)=0,$$

we obtain the following expression for the correlation function of a stationary normal random process which has passed through a linear detector:

$$B(\tau)=\frac{\sigma^2}{2\pi}\left[1+\frac{\pi}{2}R(\tau)+\frac{R^2(\tau)}{2}+\sum_{n=2}^{\infty}\frac{[(2n-3)!!]^2}{(2n)!}R^{2n}(\tau)\right]. \qquad (7.31)$$

Series (7.31) may be summed up, and then the expression of the correlation function is represented in the final form:

$$B(\tau)=\frac{\sigma^2}{2\pi}\left\{\left[\frac{\pi}{2}+\arcsin R(\tau)\right]R(\tau)+\sqrt{1-R^2(\tau)}\right\}. \qquad (7.32)$$

By expanding (7.32) into a series, it is not difficult to verify that it coincides with (3.71). As can be seen from (7.32), the difference between the correlation functions of the processes at the output and input of a linear detector is an even function of the correlation coefficient of the input process [correspondingly series (7.31) contains, besides the first power of the correlation coefficient, only even powers of $R(\tau)$] *.

If the power spectrum of a stationary normal process is concentrated in a relatively narrow frequency band around the high frequency ω_0 in such a manner, that for the correlation coefficient $R(\tau)$ of this process formula (7.15) is valid, then, in accordance with (7.24) and (7.30),

$$B(\tau)=B_0(\tau)+\frac{\sigma^2}{4}R_0(\tau)\cos\omega_0\tau+\sum_{r=1}^{\infty}B_{2r}(\tau)\cos 2r\omega_0\tau, \qquad (7.33)$$

or

$$B_0(\tau)=\frac{\sigma^2}{2\pi}\left\{1+\frac{R_0^2(\tau)}{4}+\sum_{n=2}^{\infty}\frac{[(2n-3)!!]^2\binom{2n}{n}}{(2n)!\,2^{2n}}R_0^{2n}(\tau)\right\}. \qquad (7.34)$$

$$B_{2r}=\frac{\sigma^2}{2\pi}\sum_{n=r}^{\infty}\frac{[(2n-1)!!]^2\binom{2n}{n-r}}{(2n)!\,2^{2n-1}}R_0^{2n}(\tau). \qquad (7.35)$$

---

* The fact that the correlation function of a process at the input of a linear detector does not contain any other odd powers of R except the first, is not unexpected. In fact, (7.27) may be represented in the form of the sum of two functions

$$f(x) = \tfrac{1}{2}x + f_1(x),$$

where $f_1(x) = f(x) - \frac{1}{2}x$ will be an even function of the argument x.

The first term $B_0(\tau)$ in expression (7.33) corresponds to the direct component and to the low-frequency portion of the continuous power spectrum of a random process at the output of a linear detector. Series (7.34), which represents function $B_0(\tau)$, may be summed up, and its sum may then be expressed in terms of full elliptical integrals of the first $K(R_0)$ and second $E(R_0)$ kinds (cf., e.g., N. Yanke and F. Emde. Tables of functions. Gostekhizdat, 1948)

$$B_0(\tau) = \frac{\sigma^2}{\pi^2}[2E(R_0) - (1 - R_0^2)K(R_0)]. \tag{7.36}$$

After a Fourier transformation of (7.34) and (7.36), we obtain the low-frequency portion of the power spectrum of a process at the output of a linear detector. The first term in expansion (7.34) yields the direct component, and the sum of the Fourier transformations of the even powers of the correlation coefficient yields the continuous spectrum.

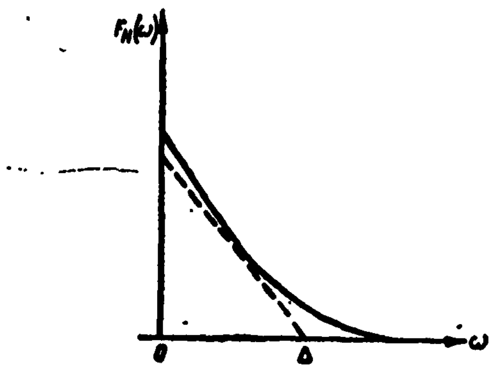


Fig. 50. Low-frequency spectrum of random process which has passed through a linear detector.

Figure 50 shows the continuous low-frequency spectrum of a process at the output of a linear detector, for the case when the power spectrum of a normal stationary process at the input is uniform in a band whose width is equal to Δ. The exponential series for R_0 in (7.34) converges so rapidly, that in practice for the computation of the spectrum it is possible to restrict one's self to only the term R_0^2. Then for the case under consideration the low-frequency portion of the continuous spectrum will have the form of a right triangle with a base of Δ. This approximate spectrum is designated in Figure 50 by the dotted line (compare Fig. 49).

Comparison with the exact spectrum indicates an entirely satisfactory approximation. The relationship of the areas of the continuous exact and approximate spectra (i.e., of power, concentrated in the low-frequency range) is equal to $\frac{\left(2-\frac{\pi}{2}\right)\frac{\sigma^2}{\pi^2}}{\sigma^2,8\pi}=1,1,$ and the spectrum density, when $\omega = 0$, (i.e., the correlation time) is 6% greater for the exact than for the approximate spectrum. In distinction from the approximate one, the exact spectrum contains frequencies higher than Δ, but their intensity is negligibly small.

The second term $\frac{\sigma^2}{4} R_0(\tau) \cos\omega_0\tau$ in expression (7.33) corresponds to the undistorted (with accuracy to the constant multiplier) reproduction, at the output of a linear detector, of the spectrum of a stationary normal random process.

The succeeding terms $B_{2r}(\tau) \cos 2r\omega_0\tau$ in expression (7.33) correspond to the high-frequency portions of the power spectrum of a process at the output of a linear detector, which lie about the even harmonics of frequency ω_0. The correlation time and, consequently, the spectrum densities when $\omega = 2r\omega_0$ diminish sharply with a rise in the harmonic number 2r, since in the expression $B_{2r}(\tau)$ [cf. (7.35)] the least exponent of $R_0(\tau)$ is equal to 2r. The areas of the continuous spectra (i.e., the powers) situated about the harmonics of ω_0, diminish in inverse proportion to the magnitude $\Gamma^2(2r + 3/2)$.

4. Approximation of Non-linear Characteristics by Exponential Series

If the function f(x), which provides an analytic concept of the characteristic of a nonlinear system, is continuous together with its derivatives, then it can be resolved into Maclaurin series

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots+\frac{x^n}{n!}f^{(n)}(0)+\ldots \qquad (7.37)$$

For this reason the nonlinear characteristic f(x) is frequently (for instance, in cases of full-wave detection) approximated by an exponential series of the type of

$$f(x)=a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots, \qquad (7.38)$$

the coefficients of which must be equal to the corresponding coefficients of series (7.37).

With such an approximation it is not difficult in the general case to determine the coefficients c_{1n} and c_{2n} in series (7.8). These coefficients are obtained from integral (7.7), which in the case under consideration has the form of*

$$c_n = \sum_{\nu=0}^{\infty} \frac{a_\nu \sigma^\nu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^\nu H_n(x-\alpha) e^{-\frac{(x-\alpha)^2}{2}} dx. \tag{7.39}$$

The integrals in (7.39) are easily computed, if the integrand function be represented as a derivative with respect to the parameter α. Then

$$c_n = \sum_{\nu=0}^{\infty} a_\nu \sigma^\nu \frac{d^n}{d\alpha^n} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^\nu e^{-\frac{(x-\alpha)^2}{2}} dx \right) =$$

$$= \sum_{\nu=0}^{\infty} a_\nu \sigma^\nu \frac{d^n}{d\alpha^n} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (x+\alpha)^\nu e^{-\frac{x^2}{2}} dx \right) =$$

$$= \sum_{\nu=0}^{\infty} \sum_{k=0}^{\nu} a_\nu \sigma^\nu \binom{\nu}{k} m_{\nu-k} \frac{d^n \alpha^k}{d\alpha^n},$$

where $m_{\nu-k}$ is a distribution moment of the $(\nu - k)$th order of a normal distribution with unitary dispersion and a zero mean. Differentiating with respect to α and employing (3.80), we find

$$c_n = \sum_{r=0}^{\infty} \sum_{k=n}^{\infty} a_{k+r} \sigma^{k+r} \binom{k+r}{k} \frac{k!}{(k-n)!} m_r \alpha^{k-n} =$$

$$= \sum_{r=0}^{\infty} \sum_{k=n}^{\infty} a_{k+2r} \sigma^{k+2r} \binom{k+2r}{k} \frac{k!}{(k-n)!} (2r-1)!! \, \alpha^{k-n},$$

or, after changing $k = n + s$ in the summation index

$$c_n = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} a_{n+s+2r} \sigma^{n+s+2r} \binom{n+s+2r}{n+s} \frac{(n+s)!\,(2r-1)!!\,\alpha^s}{s!}. \tag{7.40}$$

---

* In practice, a summation with respect to ν will contain only a small number of terms. This will indicate that the coefficients a_ν, starting with a certain ν, turn to zero. For the sake of generality we retain the summation for all positive instances of ν.

It is necessary to bear in mind, that when r = 0 it is conventional that $(2r - 1)!! = 1$. It is further necessary to average in time the product of $c_{1n} \cdot c_{2n}$, upon which depend $\alpha_1(t)$ and $\alpha_2(t)$. Designating by

$$b_{s_1, s_2}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \alpha_1^{s_1}(t) \alpha_2^{s_2}(t+\tau) dt, \tag{7.41}$$

we find from (7.10), (7.11) and (7.40) the averaged correlation function of a process at the output of a nonlinear system

$$\begin{aligned} B^*(\tau) = \sum_{n=0}^{\infty} \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \sum_{s_1=0}^{\infty} \sum_{s_2=0}^{\infty} a_{n+s_1+2r_1} a_{n+s_2+2r_2} \times \\ \times \sigma^{2n+s_1+s_2+2(r_1+r_2)} \binom{n+s_1+2r_1}{n+s_1} \binom{n+s_2+2r_2}{n+s_2} \times \\ \times \frac{(n+s_1)!(n+s_2)!(2r_1-1)!!(2r_2-1)!!}{s_1! s_2! n!} b_{s_1, s_2}(\tau) R^n(\tau). \end{aligned} \tag{7.42}$$

To the discrete portion of the power spectrum correspond (in the sense of a Fourier transformation) the terms where n = 0, and to the continuous portion, the terms where n > 0.

If the determined portion of a normal process is lacking, then in (7.42) all the terms disappear, with the exception of those obtained when $s_1 = s_2 = 0$. Then from (7.42) we find the following expression for the correlation function of a stationary, normal random process which has passed through a system, the nonlinear characteristic of which is approximated by the exponential series (7.38)

$$\begin{aligned} B(\tau) = \sum_{n=0}^{\infty} R^n(\tau) \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} a_{n+2r_1} a_{n+2r_2} \sigma^{2n+2r_1+2r_2} \times \\ \times \binom{n+2r_1}{n} \binom{n+2r_2}{n} (2r_1-1)!!(2r_2-1)!! n!, \end{aligned}$$

and since the summation along r_1 and r_2 is separable, therefore

$$B(\tau) = \sum_{n=0}^{\infty} n! R^n(\tau) \left[\sum_{r=0}^{\infty} a_{n+2r} \sigma^{n+2r} \binom{n+2r}{n} (2r-1)!! \right]^2. \tag{7.43}$$

Let us write out several of the first terms of summation (7.43), neglecting the approximating coefficients a_n higher than n = 5

$$
\begin{aligned}
B(\tau) = {} & [a_0 + a_2\sigma^2 + 3a_4\sigma^4 + \ldots]^2 + \\
& + R(\tau)[a_1\sigma + 3a_3\sigma^3 + 15a_5\sigma^5 + \ldots]^2 + \\
& + 2R^2(\tau)[a_2\sigma^2 + 6a_4\sigma^4 + \ldots]^2 + 6R^3(\tau)[a_3\sigma^3 + 10a_5\sigma^5 + \ldots]^2 + \\
& + 24R^4(\tau)[a_4\sigma^4 + \ldots]^2 + 120R^5(\tau)[a_5\sigma^5 + \ldots]^2.
\end{aligned}
\tag{7.44}
$$

In expression (7.44), the first line yields the power of the direct component, the second line corresponds to the undistorted reproduction at the output of a nonlinear system of the input power spectrum, and the succeeding terms are equal to the products of the second-, third-, and higher-order nonlinear distortions of this spectrum.

Let us investigate in greater detail the case when the characteristic of a nonlinear system is approximated by a parabola

$$y = a_0 + a_1x + a_2x^2. \tag{7.45}$$

In this case from (7.40) we find

$$
\begin{aligned}
c_0 &= a_0 + a_2\sigma^2 + a_1\sigma\alpha + a_2\sigma^2\alpha^2, \\
c_1 &= a_1\sigma + 2a_2\sigma^2\alpha, \\
c_2 &= 2a_2\sigma^2, \\
c_n &\equiv 0 \text{ when } n \geq 3,
\end{aligned}
\tag{7.46}
$$

where $\alpha(t) = \dfrac{S(t)}{\sigma}$.

Let us introduce the designations

$$A_s = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} S(t)\,dt, \tag{7.47}$$

$$W_s = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} S^2(t)\,dt, \tag{7.48}$$

$$B_s(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} S(t)S(t+\tau)\,dt, \tag{7.49}$$

$$B_{s,s^2}(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} S(t)S^2(t+\tau)\,dt, \tag{7.50}$$

$$B_{s^2}(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} S^2(t)S^2(t+\tau)\,dt. \tag{7.51}$$

The magnitude A_s represents the direct component of the process, and the magnitude W_s its mean power. The magnitudes $B_s(\tau)$, $B_{s^2}(\tau)$ and $B_{s,s^2}(\tau)$ are respectively the autocorrelation functions of process $S(t)$, of its square $S^2(t)$ and the mutual correlation function of $S(t)$ and $S^2(t)$.

With the employment of the designations introduced above, the expression of the averaged correlation function of the random process obtained as a result of the quadratic transformation of a normal process, may be represented in the form [cf. (7.42)] of

$$\begin{aligned} B^*(\tau)=&(a_0+a_2\sigma^2)(a_0+a_2\sigma^2+2a_1A_s+2a_2W_s)+\\ &+a_1^2B_s(\tau)+2a_1a_2B_{s,s^2}(\tau)+a_2^2B_{s^2}(\tau)+\\ &+\sigma^2[a_1^2+4a_1a_2A_s+4a_2^2B_s(\tau)]R(\tau)+2a_2^2\sigma^4R^2(\tau). \end{aligned} \tag{7.52}$$

Each item in (7.52) has a clear physical interpretation. The first line yields the power of the direct component, the second line corresponds to the discrete portion of the spectrum, and the last line to the continuous portion of the spectrum.

The direct component contains elements both of the determined and of the random parts of the process at the input, the share of the determined part of the process in the direct component being equal to $2a_0(a_1A_s + a_2W_s)$, and the share of the random part coming to $(a_0 + a_2\sigma^2)^2$. In addition, the direct component also contains the mean power of the beat between the components of the determined and the random parts of a normal process. This latter is equal to $2a_2\sigma^2(a_1A_s + a_2W_s)$.

The discrete spectrum after quadratic transformation reproduces the discrete input spectrum [the term $a_1^2B_s(\tau)$], and also contains combination harmonics of the mutual

beats of the components of the determined part of a normal process [the succeeding terms in the second line of formula (7.52)].

The continuous spectrum after quadratic transformation reproduces the input continuous spectrum [the term $a_1^2 \sigma^2 R(\tau)$], and also contains combination harmonics of the mutual beats of the components of the random part [the term $2a_2^2 \sigma^4 R^2(\tau)$] and of the components of the determined and the random parts [the remaining terms in the last line of formula (7.52)].

5. Square-law Detection of an Amplitude-modulated Signal in the Presence of Noise.

Let us assume, that the determined part of a normal process constitutes an amplitude-modulated signal

$$S(t) = u(t) \cos \omega_0 t, \tag{7.53}$$

the highest harmonic in the spectrum of the envelope u(t) being much smaller than the carrier frequency ω_0.

Let us assume that the stationary random part of the normal process represents noise, the power spectrum of which is concentrated in a relatively narrow frequency band about the same high frequency ω_0. Then the correlation coefficient $R(\tau)$ of the noise may be represented in the form of (7.15).

Let us employ the results of the preceding section for solving the problem of the detection of an amplitude-modulated signal in the presence of noise. It is obvious that, in order to restore the low-frequency envelope u(t) from the radio signal, the detector must contain, in addition to a nonlinear element, a filtering element which separates out the low-frequency components and suppresses the high-frequency ones.

Let us first examine the nonlinear transformation of the signal with the noise, assuming for the sake of a simple illustration of the general results, that as the nonlinear characteristic of the detector there serves the parabola $y = a_2 x^2$. (The coefficient a_2 may be assumed equal to unity, since it serves the purpose of scale

and may always be taken into account in the final results).

From (7.52) when $a_0 = a_1 = 0$ and $a_2 = 1$, we find

$$B^{*}(\tau)=\sigma^4+2\sigma^2 W_s+B_{s^2}(\tau)+4\sigma^2 B_s(\tau)R(\tau)+2\sigma^4 R^2(\tau). \tag{7.54}$$

We now determine the magnitudes W_s, B_s and $B_{s}2$ for an amplitude-modulated signal. Substituting (7.53) into (7.48), (7.49) and (7.51), we obtain respectively

$$W_s=\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}u^2(t)\cos^2\omega_0 t\,dt=$$

$$=\frac{1}{2}\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}u^2(t)\,dt+\frac{1}{2}\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}u^2(t)\cos 2\omega_0 t\,dt; \tag{7.55}$$

$$B_s(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}u(t)\,u(t+\tau)\cos\omega_0 t\cos\omega_0(t+\tau)\,dt=$$

$$=\frac{\cos\omega_0\tau}{2}\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}u(t)\,u(t+\tau)\,dt+$$

$$+\frac{1}{2}\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}u(t)\,u(t+\tau)\cos 2\omega_0\left(t+\frac{\tau}{2}\right)dt; \tag{7.56}$$

$$B_{s^2}(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}u^2(t)\,u^2(t+\tau)\cos^2\omega_0 t\cos^2\omega_0(t+\tau)\,dt=$$

$$=\frac{1}{4}\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}u^2(t)\,u^2(t+\tau)\,[1+\cos 2\omega_0 t+\cos 2\omega_0(t+\tau)+$$

$$+\cos 2\omega_0 t\cos 2\omega_0(t+\tau)]\,dt=\frac{1}{4}\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}u^2(t)\,u^2(t+\tau)\,dt+$$

$$+\frac{1}{4}\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}u^2(t)\,u^2(t+\tau)\,[\cos 2\omega_0 t+\cos 2\omega_0(t+\tau)]\,dt+ \tag{7.57}$$

$$+\frac{\cos 2\omega_0\tau}{8}\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}u^2(t)\,u^2(t+\tau)\,dt+$$

$$+\frac{1}{8}\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}u^2(t)\,u^2(t+\tau)\cos 4\omega_0\left(t+\frac{\tau}{2}\right)dt. \tag{7.57 (cont'd)}$$

It can be shown that, with the assumption made above concerning the spectrum of the signal envelope, a whole series of limits in the cited expressions turns to zero. For this use should be made of the fact, that

$$\begin{aligned}&\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\cos\omega_s t\cos\omega_n(t+\tau)\,dt=\\&=\frac{1}{2}\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\cos[(\omega_n-\omega_s)t+\omega_n\tau]\,dt+\\&+\frac{1}{2}\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\cos[(\omega_n+\omega_s)t+\omega_n\tau]\,dt=\\&=\begin{cases}\frac{1}{2}\cos\omega_n\tau\ \text{when}\ \omega_n=\omega_s,\\ 0\qquad \text{when}\ \omega_n\neq\omega_s.\end{cases}\end{aligned} \tag{7.58}$$

Then in (7.55) the second limit turns to zero and, consequently,

$$W_s=\frac{1}{2}\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}u^2(t)\,dt=\frac{W_1}{2}, \tag{7.59}$$

where W_1 is the mean power of the modulating signal u(t).

In (7.56) the second limit also turns to zero and, consequently

$$B_s(\tau)=\frac{\cos\omega_0\tau}{2}\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}u(t)\,u(t+\tau)\,dt=\frac{1}{2}B_u(\tau)\cos\omega_0\tau, \tag{7.60}$$

where $B_u(\tau)$ is the autocorrelation function of the modulating signal, with $B_u(0) = W_1$.

In (5.57) the second and fourth limits turn to zero and, consequently,

$$B_{s^2}(\tau)=\frac{1}{4}\left(1+\frac{1}{2}\cos 2\omega_0\tau\right)\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}u^2(t)\,u^2(t+\tau)\,dt= \tag{7.61}$$

$$=\frac{1}{4}\left(1+\frac{1}{2}\cos 2\omega_0\tau\right)B_{u^2}(\tau),$$

where $B_{u^2}(\tau)$ is the autocorrelation function of the square of the modulating signal.

Substituting (7.59) - (7.61) into (7.54), taking (7.15) into account, we find

$$\begin{aligned}B^{*}(\tau)&=\sigma^4+\sigma^2B_u(0)+\frac{1}{4}B_{u^2}(\tau)+\sigma^2B_u(\tau)R_0(\tau)+\\&+\sigma^4R_0^2(\tau)+\left[\frac{1}{8}B_{u^2}(\tau)+\sigma^2B_u(\tau)R_0(\tau)+\sigma^4R_0^2(\tau)\right]\cos 2\omega_0\tau,\end{aligned} \tag{7.62}$$

where $R_0(\tau)$ is the correlation coefficient of the envelope of the noise at the input of the square-law detector.

In the absence of a signal it follows from (7.62) that

$$B(\tau)=\sigma^4\left[1+R_0^2(\tau)+R_0^2(\tau)\cos 2\omega_0\tau\right]. \tag{7.63}$$

In distinction from a linear detector [cf. (7.33)], for which the output correlation function of noise is expressed as an infinite series in terms of the exponents of the input correlation coefficient $R(\tau)$, the correlation function of noise at the output of a square-law detector contains no power of $R(\tau)$ higher than the second.

Employing the expressions obtained for the correlation function and performing a Fourier transformation, it is possible to determine the corresponding power spectrum.

We shall illustrate the sequence of the computation of the power spectrum of a random process at the output of a square-law detector, by an example in which the modulating signal is harmonic, with a frequency of Ω, i.e.,

$$u(t)=u_0(1+m\cos\Omega t). \tag{7.64}$$

Let us, in addition, assume that the power spectrum of noise at the input of the detector is uniform in the band of Δ and is symmetrical with respect to the high

frequency ω_0 ($\Delta \ll \omega_0$, $\Delta > 2\Omega$). Then in accordance with (6.18) the correlating coefficient of the process at the input of the detector is equal to

$$R(\tau) = R_0(\tau)\cos\omega_0\tau = \frac{\sin\frac{\tau\Delta}{2}}{\frac{\tau\Delta}{2}}\cos\omega_0\tau. \tag{7.65}$$

To determine the correlation function of the process at the output of a square-law detector we shall employ formula (7.62). We shall first compute the magnitudes B_u and B_{u2}.

Substituting (7.64) into (7.60) and (7.61), and also taking into account (7.58), we find

$$B_u(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{-\frac{T}{2}} u_0^2(1+m\cos\Omega t)[1+m\cos\Omega(t+\tau)]\,dt = \tag{7.66}$$

$$= u_0^2\left(1+\frac{m^2}{2}\cos\Omega\tau\right),$$

$$B_u(0) = u_0^2\left(1+\frac{m^2}{2}\right). \tag{7.67}$$

$$B_{u^2}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} u_0^4(1+m\cos\Omega t)^2[1+m\cos\Omega(t+\tau)]^2\,dt =$$

$$= u_0^4\left(1+m^2+\frac{m^4}{4}+2m^2\cos\Omega\tau+\frac{m^4}{8}\cos 2\Omega\tau\right). \tag{7.68}$$

Substituting (7.66) - (7.68) into (7.62) and effecting a regrouping of the terms, we obtain

$$\begin{aligned} B^*(\tau) = {} & \frac{1}{4}\left[2\sigma^2 + u_0^2\left(1+\frac{m^2}{2}\right)\right]^2 + \frac{u_0^4}{2}m^2\cos\Omega\tau + \\ & + \frac{u_0^4 m^4}{32}\cos 2\Omega\tau + \frac{u_0^4}{8}\left(1+\frac{m^2}{2}\right)^2\cos 2\omega_0\tau + \frac{u_0^4 m^2}{4}\cos\Omega\tau\cos 2\omega_0\tau + \\ & + \frac{u_0^4 m^4}{64}\cos 2\Omega\tau\cos 2\omega_0\tau + u_0^2\sigma^2 R_0(\tau)\left(1+\frac{m^2}{2}\cos\Omega\tau\right) + \\ & + \sigma^4 R_0^2(\tau) + u_0^2\sigma^2 R_0(\tau)\left(1+\frac{m^2}{2}\cos\Omega\tau\right)\cos 2\omega_0\tau + \\ & + \sigma^4 R_0^2(\tau)\cos 2\omega_0\tau, \end{aligned} \tag{7.69}$$

with $R_0(\tau)$ being determined from (7.65).

To compute the power spectrum of a process at the output of a square-law detector it is now necessary, in accordance with Khinchin's formula (5.44), to effect

an inverse Fourier transformation on $B^*(\tau)$. From the expression of the correlation function (7.69) it is evident, that this spectrum consists of two parts: discrete and continuous. The first terms of (7.69), not containing $R_0(\tau)$, after the Fourier transformation yield delta-functions, i.e., these terms correspond to the discrete part of the power spectrum of the process. The terms with $R_0(\tau)$ correspond to the continuous part of the spectrum, the terms containing the multiplier $\cos 2\omega_0\tau$ corresponding to the high-frequency range of the continuous spectrum, and the rest to its low-frequency range.

Let us designate by $F_{\text{Д}}(\omega)$ the discrete part of the power spectrum of a process at the output of a square-law detector. Then from (7.69) we obtain*

$$\begin{aligned}\frac{1}{2\pi}F_{\text{Д}}(\omega)=&\frac{1}{4}\left[2\sigma^2+u_0^2\left(1+\frac{m^2}{2}\right)\right]^2\delta(\omega)+\frac{u_0^4m^2}{2}\delta(\omega-\Omega)+\\&+\frac{u_0^4m^4}{32}\delta(\omega-2\Omega)+\frac{u_0^4}{8}\left(1+\frac{m^2}{2}\right)^2\delta(\omega-2\omega_0)+\\&+\frac{u_0^4m^2}{8}[\delta(\omega-2\omega_0-\Omega)+\delta(\omega-2\omega_0+\Omega)]+\\&+\frac{u_0^4m^4}{128}[\delta(\omega-2\omega_0-2\Omega)+\delta(\omega-2\omega_0+2\Omega)].\end{aligned}\tag{7.70}$$

The first term in (7.70) corresponds to the direct component of the process at the output of the detector. The share of the signal in the direct component is equal to $\frac{u_0^4}{4}\left(1+\frac{m^2}{2}\right)^2$, and the share of noise is σ^4.

Besides this, the mean power of the beat between the harmonic components of the signal and the noise equal to $\sigma^2 u_0^2\left(1+\frac{m^2}{2}\right)$ enters into the composition of the direct component. The remaining terms of the discrete power spectrum (7.70) owe their existence to the mutual beats of the harmonic components of the signal in its passage through the square-law detector.

We pass to the determination of the continuous power spectrum. Let us designate its low-frequency and high-frequency parts by $F_H(\omega)$ and $F_B(\omega)$, respectively. From (7.69) we find

---

* To each $\cos\omega_i\tau$ corresponds a semi-sum of the delta-functions (cf. Appendix IV). Here are written out only those delta-functions, whose arguments turn to zero with positive frequencies. For preservation of the power relationships the coefficients of these functions are doubled.

$$F_u(\omega)=4u_0^2\sigma^2\int_0^\infty R_0(\tau)\cos\omega\tau d\tau+$$

$$+2u_0^2 m^2\sigma^2\int_0^\infty R_0(\tau)\cos\Omega\tau\cos\omega\tau d\tau+4\sigma^4\int_0^\infty R_0^2(\tau)\cos\omega\tau d\tau= \tag{7.71}$$

$$=F_1(\omega)+F_2(\omega)+F_3(\omega).$$

We compute each item in (7.71) separately. Since according to our assumption the spectrum of the noise at the input is uniform in the band Δ, therefore

$$F_1(\omega)=4u_0^2\sigma^2\int_0^\infty R_0(\tau)\cos\omega\tau d\tau=\begin{cases}\frac{4\pi}{\Delta}\sigma^2u_0^2, & 0<\omega<\frac{\Delta}{2},\\ 0, & \omega>\frac{\Delta}{2}.\end{cases} \tag{7.72}$$

Spectrum $F_1(\omega)$ is uniform in a low-frequency band with a width of Δ, (left part of Fig. 51). Employing (7.72) and bearing in mind the condition that $\Delta>2\Omega$, we find

$$F_2(\omega)=2u_0^2m^2\sigma^2\int_0^\infty R_0(\tau)\cos\Omega\tau\cos\omega\tau d\tau=$$

$$=u_0^2m^2\sigma^2\int_0^\infty R_0(\tau)\cos(\omega+\Omega)\tau d\tau+$$

$$+u_0^2m^2\sigma^2\int_0^\infty R_0(\tau)\cos(\omega-\Omega)\tau d\tau=$$

$$=\frac{m^2}{4}[F_1(\omega+\Omega)+F_1(\omega-\Omega)],$$

or

$$F_2(\omega)=\begin{cases}\frac{2\pi}{\Delta}\sigma^2m^2u_0^2, & 0<\omega<\frac{\Delta}{2}-\Omega,\\ \frac{\pi}{\Delta}\sigma^2m^2u_0^2, & \frac{\Delta}{2}-\Omega<\omega<\frac{\Delta}{2}+\Omega,\\ 0, & \omega>\frac{\Delta}{2}+\Omega.\end{cases} \tag{7.73}$$

Spectrum $F_2(\omega)$ is shown in the center of Figure 51.

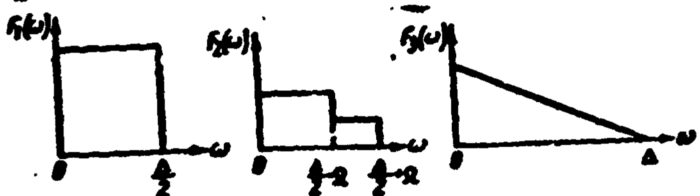


Fig. 51. Components of continuous spectrum.

Let us finally compute the third item, $F_3(\omega)$. For this we employ the relationship:* if $f(\tau)$ and $\phi(\omega)$ are a pair of Fourier transformations, then

$$\int_0^{\infty} f^2(\tau)\cos\omega\tau d\tau = \frac{1}{4\pi}\int_{-\infty}^{\infty}\Phi(u)\Phi(\omega-u)\,du. \qquad (7.74)$$

Assuming in (7.74) $f(\tau) = 2\sigma^2 R_0(\tau)$ and $\phi(\omega) = \frac{4\pi\sigma^2}{\Delta}$ when $0<\omega<\frac{\Delta}{2}$, $\phi(\omega)\equiv 0$ when $\omega > \frac{\Delta}{2}$, we obtain

$$F_3(\omega) = 4\sigma^4\int_0^{\infty} R_0^2(\omega)\cos\omega\tau d\tau = \frac{1}{4\pi}\int_{-\frac{\Delta}{2}+\omega}^{\frac{\Delta}{2}}\left(\frac{4\pi\sigma^2}{\Delta}\right)^2 d\omega =$$
$$= \begin{cases} \frac{4\pi\sigma^4}{\Delta^2}(\Delta-\omega), & 0<\omega<\Delta, \\ 0, & \omega>\Delta. \end{cases} \qquad (7.75)$$

Spectrum $F_3(\omega)$ has the form of a right triangle with a base the length of Δ (Fig. 51, right). Summing up $F_1(\omega)$, $F_2(\omega)$ and $F_3(\omega)$, we find the low-frequency part of the continuous spectrum of a process at the output of a square-law detector (Fig. 52, a).

It can be seen from (7.69), that the terms corresponding to the high-frequency part of a continuous spectrum differ from the corresponding terms of its low-frequency part only by the multiplier $\cos 2\omega_0\tau$. Therefore $F_B(\omega)$ is obtained by the shift of $F_H(\omega)$ into the high-frequency range by $2\omega_0$ and by its multiplication by a constant, equal to 1/2 (Fig. 52, b).

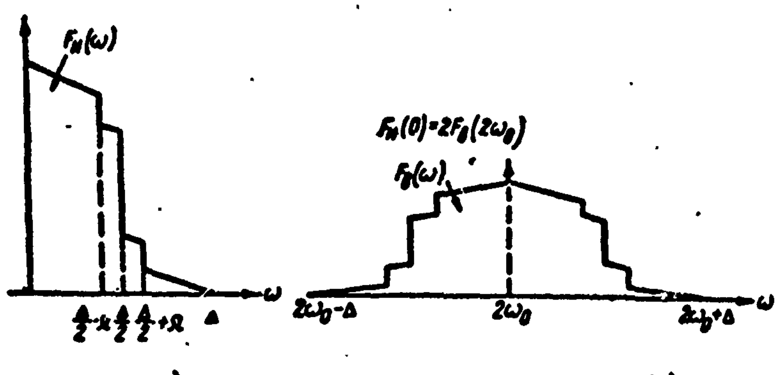


Fig. 52. Continuous spectrum of process at output of square-law detector.
a) low frequency part, b) high-frequency part.

---

* This relationship is an analytic notation of the convolution theorem (compare p 222).

The full power of the continuous part of the spectrum, as can be seen from Figure 52, is evenly distributed between the low-frequency and the high-frequency ranges.

Components $F_1(\omega)$ and $F_2(\omega)$ of the continuous spectrum and the corresponding items in the high-frequency range are dependent on the beats of the harmonic components of the signal with the components of the noise at the input of the detector, which fill band Δ. This part of the continuous spectrum at the output of a detector is sometimes called the continuous spectrum of signal-noise cross-modulation.

Component $F_3(\omega)$ of the continuous spectrum and the corresponding item in the high-frequency range are dependent on the beats of any two elementary noise components at the input of the detector, the low-frequency component being obtained on the basis of subtractive combination harmonics, and the high-frequency component on the basis of additive harmonics. Since, with a fixed band Δ at the input, the number of pairs of harmonic components with a definite frequency difference diminishes as this difference is increased, the intensity of the spectrum under consideration diminishes, this diminution taking place according to the linear law for the rectangular form of the input spectrum. The spectrum $F_3(\omega)$ depicted on the right side of Figure 51 obviously coincides with the low-frequency spectrum of the process at the output of a square-law detector, on the input of which there acts only a stationary random process (noise in the absence of a signal).

Comparing this spectrum with the spectrum shown by the broken line in Figure 50, we conclude that the low-frequency continuous noise spectra, at the output of a linear and of a square-law detector, in the first approximation coincide with one another. It is clear that, if the action of the filtering element of the detector is taken into account, the high-frequency components will be suppressed.

The ratio of the power of cross-modulation to the power of the noise is equal to

$$\frac{\int_0^\infty [F_1(\omega) + F_2(\omega)]\, d\omega}{\int_0^\infty F_3(\omega)\, d\omega} = \left(\frac{u_0}{\sigma}\right)^2 \left(1 + \frac{m^2}{2}\right), \qquad (7.76)$$

i.e., to the ratio of the mean power of the signal to the full power of the noise at the input of the detector. Therefore with a strong signal ($u_0 \gg \sigma$) the principal part of the continuous spectrum at the output of a square-law detector has to do with signal-noise cross modulation, and with a weak signal ($u_0 \ll \sigma$) the principal part is played by the continuous noise spectrum. In accordance with this, with a strong signal the form of the power spectrum approaches that of a step-shaped curve with slightly sloped steps (Fig. 53, left), and with a weak signal the spectral picture resembles "a triangle with a saw-tooth hypotenuse" (sic) (Fig. 53, right).

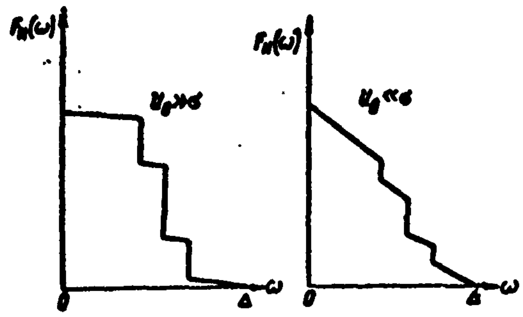


Fig. 53. Comparison of power spectra with strong and weak signals.

In the general case of an arbitrary amplitude-modulated signal, the structure of the power spectrum at the output of a square-law detector will be analogous. The terms of correlation function (7.54) which do not depend on $R(\tau)$, will yield in the spectrum a direct component and discrete components from the mutual beats of the harmonic components of the signal. The terms in (7.54) proportional to $R(\tau)$ will yield the continuous cross-modulation spectrum, and the terms proportional to $R^2(\tau)$ will yield the continuous spectrum resulting from the passage through the detector of noises alone.

6. Solution Obtained by Contour Integral Method.

In many cases the nonlinear characteristic is approximated by the function

$$\begin{aligned} f(x) &= a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots, \quad x > x_0, \\ f(x) &\equiv 0, \quad x < x_0, \end{aligned} \tag{7.77}$$

where X_0 is the cutoff voltage.

An expression for the power spectrum of a normal random process, which has passed through a nonlinear system with a characteristic of (7.77), may be obtained by the method set forth in Sect. 1. The indicated method is successfully used in the investigation of the half-wave detection of a stationary normal random process (cf. sect. 3). However, if the normal process contains also a determined part, then difficulties may arise in the time averaging procedure according to formula (7.9). In order to facilitate the solution of the problem, it may prove useful to employ the method of contour integrals, bearing in mind that function (7.77) permits a representation in the form of (6.62)*.

Let us then assume, that the characteristic $y = f(x)$ of a nonlinear system permits representation by the contour integral (6.62) and let us return to the general expression (7.2) for the correlation function of the random process at the output of a nonlinear system, at the input of which there acts a normal random process.

Substituting (6.62) into (7.2) and changing the order of integration, we find

$$B(\tau, t) = \frac{1}{4\pi^2}\int_{C_1} g(iu_1)\int_{C_2} g(iu_2)\frac{1}{2\pi\sigma^2\sqrt{1-R^2}}\times$$
$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{(x_1-a_1)^2-2R(x_1-a_1)(x_2-a_2)+(x_2-a_2)^2}{2\sigma^2(1-R^2)}} e^{i(x_1u_1+x_2u_2)}dx_1dx_2du_1du_2. \tag{7.78}$$

The double integral along x_1 and x_2 represents a two-dimensional characteristic function of normal distribution. Therefore, employing (3.95), we obtain

$$B(\tau, t) = \frac{1}{4\pi^2}\int_{C_1}\int_{C_2} g(iu_1)\, g(iu_2) e^{i(a_1u_1+a_2u_2)} e^{-\frac{\sigma^2}{2}\left(u_1^2+2Ru_1u_2+u_2^2\right)} du_1du_2. \tag{7.79}$$

In integral (7.79) only the multiplier $e^{i(a_1u_1+a_2u_2)}$ contains the magnitudes $a_1 = s(t)$ and $a_2 = s(t+\tau)$, which depend upon time. Therefore in the time averaging of the correlation function $B(\tau, t)$ only this multiplier is averaged. Designating

$$\Phi_s(u_1, u_2, \tau) = \lim_{T\to\infty}\frac{1}{T}\int_0^T e^{i[s(t)u_1+s(t+\tau)u_2]}dt, \tag{7.80}$$

* Of course, characteristic (7.38) may be represented in the form of a sum of two characteristics of the (7.77) type: $f_1(x) + f_2(-x)$, and then the method of contour integrals may be expanded to full-wave detection.

we find from (7.79) the following expression of the averaged correlation function $B^*(\tau)$ of a random process at the output of a nonlinear system

$$B^*(\tau)=\frac{1}{4\pi^2}\int_{c_1}\int_{c_2} g(iu_1)g(iu_2)\theta_s(u_1,u_2,\tau)e^{-\frac{\sigma^2}{2}\left(u_1^2+2Ru_1u_2+u_2^2\right)}du_1du_2. \quad (7.81)$$

Let us assume that the determined part of the process constitutes a periodic function of time with a period of T. Approximating the periodic function S(t) by the first N + 1 terms of the Fourier series

$$S(t)=\sum_{n=0}^{N}A_n\cos\frac{2\pi n}{T}t$$

and noting that the averaging in (7.80) may in this case be made for one period, we obtain the following expression of the function $\theta_s(u_1, u_2, \tau)$

$$\theta_s(u_1, u_2, \tau)=\frac{1}{T}\int_0^T e^{i\sum\limits_{n=0}^{N}\left[u_1A_n\cos\frac{2\pi n}{T}t+u_2A_n\cos\frac{2\pi n}{T}(t+\tau)\right]}dt,$$

or, introducing a new variable $v = \frac{2\pi t}{T}$ and separating out the direct component A_0, we find

$$\theta_s(u_1, u_2, \tau)=e^{iA_0(u_1+u_2)}\frac{1}{2\pi}\int_0^{2\pi} e^{i\sum\limits_{n=1}^{N}\left[u_1A_n\cos nv+u_2A_n\cos\left(nv+\frac{2\pi n\tau}{T}\right)\right]}dv. \quad (7.82)$$

We perform under the summation sign in the integrand function the elementary transformations

$$u_1\cos nv+u_2\cos\left(nv+\frac{2\pi n\tau}{T}\right)=\left(u_1+u_2\cos\frac{2\pi n\tau}{T}\right)\cos nv-$$
$$-u_2\sin\frac{2\pi n\tau}{T}\sin nv=\sqrt{u_1^2+u_2^2+2u_1u_2\cos\frac{2\pi n\tau}{T}}\cos(nv+\varphi_n),$$

wherein the phase angle φ_n does not depend on v. The expression (7.82) may now be rewritten in the form of

$$\theta_s(u_1, u_2, \tau)=$$
$$=e^{iA_0(u_1+u_2)}\frac{1}{2\pi}\int_0^{2\pi} e^{i\sum\limits_{n=1}^{N}A_n\sqrt{u_1^2+u_2^2+2u_1u_2\cos\frac{2\pi n\tau}{T}}\cos(nv+\varphi_n)}dv. \quad (7.83)$$

The integral in (7.83) is a zero-order Bessel function of N variables

$$\theta_s(u_1, u_2, \tau) = J_0(x_1, x_2, \dots x_N)\, e^{iA_0(u_1+u_2)},$$
$$x_n = A_n\sqrt{u_1^2 + u_2^2 + 2u_1u_2\cos\frac{2\pi n\tau}{T}}\ (n = 1, 2, \dots N). \tag{7.84}$$

A Bessel function of N variables may be expanded into a short series of ordinary Bessel functions*, then

$$\theta_s(u_1, u_2, \tau) = e^{iA_0(u_1+u_2)} \times$$
$$\times \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} J_{-k_1}(x_1)\, J_{k_2}(x_2) \dots J_{k_N}(x_N), \tag{7.85}$$

where $k_1 = 2\ k_2 + \dots + Nk_n$.

Let us examine more closely the case of the purely harmonic signal $S(t)=a\cos\frac{2\pi t}{T}$. In this case from (7.85) we find

$$\theta_s(u_1, u_2, \tau) = J_0(x_1) = J_0\left(a\sqrt{u_1^2 + u_2^2 + 2u_1u_2\cos\frac{2\pi\tau}{T}}\right). \tag{7.86}$$

Employing the addition theorem, well-known in the theory of Bessel functions (cf., e.g. G. N. Vatson, "Teoriya besselevskikh funktsiy", For. Lit. Pub. Hse. 1949, p. 391), [i.e., G. N. Watson, "A Treatise on the Theory of Bessel Functions", 2nd ed., New York, 1944]), it is possible to represent (7.86) in the form of the series

$$\theta_s(u_1, u_2, \tau) = \sum_{n=0}^{\infty} (-1)^n \varepsilon_n J_n(au_1)\, J_n(au_2) \cos\frac{2\pi n\tau}{T}, \tag{7.87}$$

where $\varepsilon_0 = 1$, $\varepsilon_n = 2$ when $n \geqslant 1$.

If now (7.87) is substituted into (7.81) and, besides that, there is employed an expansion of the comultiplier $e^{-\sigma^2 R u_1 u_2}$ into the series

$$e^{-\sigma^2 R u_1 u_2} = \sum_{k=0}^{\infty} (-1)^k \frac{\sigma^{2k} R^k}{k!} u_1^k u_2^k. \tag{7.88}$$

* Cf. M. I. Akimov. On the Bessel Functions of Many Variables. Leningrad, 1929, pp. 47 - 49. [O funktsiyakh besselya mnogikh peremennykh.]

then in the double integral (7.81) it will be possible to separate the variables of integration and to represent the correlation function $B^*(\tau)$ in the form [compare (6.65)] of

$$B^*(\tau)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{n+k}\varepsilon_n\frac{\sigma^{2k}R^k(\tau)}{k!}\cos\frac{2\pi n\tau}{T}\times$$

$$\times\frac{1}{2\pi}\int_{C_1}g(iu_1)u_1^kJ_n(au_1)e^{-\frac{\sigma^2u_1^2}{2}}du_1\frac{1}{2\pi}\int_{C_2}g(iu_2)u_2^kJ_n(au_2)e^{-\frac{\sigma^2u_2^2}{2}}du_2.$$

Designating

$$h_{nk}=\frac{i^{n+k}}{2\pi}\int_{C}g(iu)u^kJ_n(au)e^{-\frac{\sigma^2u^2}{2}}du, \tag{7.89}$$

we finally obtain

$$B^*(t)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\varepsilon_n\sigma^{2k}}{k!}R^k(\tau)h_{nk}^2\cos\frac{2\pi n\tau}{T}. \tag{7.90'}$$

Analogous computations for the case when the signal consists of two harmonic vibrations

$$a_1\cos\frac{2\pi}{T_1}t+a_2\cos\frac{2\pi}{T_2}t,$$

lead to the formula

$$B^*(\tau)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\varepsilon_m\varepsilon_n\sigma^{2k}}{k!}R^k(\tau)h_{mnk}^2\cos\frac{2\pi m}{T_1}t\cos\frac{2\pi n}{T_2}t, \tag{7.90}$$

where

$$h_{mnk}^2=\frac{i^{m+n+k}}{2\pi}\int_{C}g(iu)u^kJ_m(a_1u)J_n(a_2u)e^{-\frac{\sigma^2u^2}{2}}du.$$

Let us assume that the power spectrum of the stationary part of the process is concentrated in the narrow frequency band about the carrier frequency $\omega_0=\frac{2\pi}{T}$ of the signal. Then the correlation coefficient may be represented in the form of $R(\tau)=R_0(\tau)\cos\omega_0\tau$ [cf. (7.15)], and from (7.90) we obtain

$$B^*(\tau)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\varepsilon_n\sigma^{2k}}{k!}R_0^k(\tau)h_{nk}^2\cos^k\omega_0\tau\cos n\omega_0\tau. \tag{7.91}$$

Replacing the powers of the cosines by the sum of the cosines of the multiple arcs according to formula (7.17) and (7.18) and effecting the same transformations as in §2, it is not difficult to transform (7.91) into an expression analogous to (7.24):

$$\begin{aligned} B^*(\tau) = \sum_{n=0}^{\infty} \varepsilon_n h_{n,0}^2 \cos n\omega_0\tau + \sum_{n=0}^{\infty} \varepsilon_n B_n(\tau) \cos n\omega_0\tau + \\ + \frac{1}{2}\sum_{n=0}^{\infty}\sum_{r=1}^{\infty} \varepsilon_n B_{2r-1,n}(\tau)\left[\cos(n+2r-1)\omega_0\tau + \cos(n-2r+1)\omega_0\tau\right] + \\ + \frac{1}{2}\sum_{n=0}^{\infty}\sum_{r=1}^{\infty} \varepsilon_n B_{2r,n}(\tau)\left[\cos(n+2r)\omega_0\tau + \cos(n-2r)\omega_0\tau\right], \end{aligned} \tag{7.92}$$

where

$$B_n(\tau) = \sum_{k=1}^{\infty} \frac{\sigma^{4k}}{(2k)!} \frac{\binom{2k}{k}}{2^{2k}} h_{n,2k}^2 R_0^{2k}(\tau), \tag{7.93}$$

$$B_{2r-1,n}(\tau) = \sum_{k=r}^{\infty} \frac{\sigma^{4k-2}}{(2k-1)!} \frac{\binom{2k-1}{k-r}}{2^{2k-2}} h_{n,2k-1}^2 R_0^{2k-1}(\tau), \tag{7.94}$$

$$B_{2r,n}(\tau) = \sum_{k=r}^{\infty} \frac{\sigma^{4k}}{(2k)!} \frac{\binom{2k}{k-r}}{2^{2k-1}} h_{n,2k}^2 R_0^{2k}(\tau). \tag{7.94'}$$

The power spectrum of a process at the output of a nonlinear system is obtained by a Fourier transformation of $B^*(\tau)$. The discrete part of this spectrum corresponds to the first sum in (7.92), while the remaining terms of this equation yield the continuous part.

It can be seen from (7.90), that determination of the power spectrum of the sum of a periodic signal and a stationary normal random process reduces, after nonlinear transformation, to the computation of Fourier transformations of the exponents of the input correlation coefficient and of the integrals (7.89), which depend on the characteristic of nonlinearity g(iu). For the nonlinear characteristic

$$f(x) = \begin{cases} a_\nu (x - x_0)^\nu, & x > x_0 \\ 0, & x < x_0 \end{cases} \tag{7.95}$$

the function $g(iu) = \frac{a_\nu \Gamma(\nu+1)}{(iu)^{\nu+1}} e^{-iux_0}$, and contour c coincides with the true axis, skirting only the origin of the coordinates along a semicircle in the lower half of the plane(Fig. 54). In this case the coefficients (7.89) are expressed as integrals

of the type of

$$h_{nk}=\frac{i^{n+k-\nu-1}a_0\Gamma(\nu+1)}{2\pi}\int_c u^{k-\nu-1}J_n(au)e^{-\frac{\sigma^2u^2}{2}}e^{-ix_0u}du. \tag{7.96}$$

The integrals (7.96) may be computed by means of the expansion of e^{-ix_0u} and $J_n(au)$ into exponential series and by the replacement of $\zeta = u^2$, after which the problem reduces to the computation of the contour integrals $\int_{c^*}\zeta^{s-1}e^{-\zeta}d\zeta$ (contour c* is shown in Fig. 54), which simply coincide with the well-known integral representation of the gamma-function (cf. M. A. Lavrent'yev and B. V. Shabat. Metody teorii funktsiy kompleksnogo peremennogo (Methods of the Theory of the Functions of the Complex Variable). Gostekhizdat, 1951, p. 373).

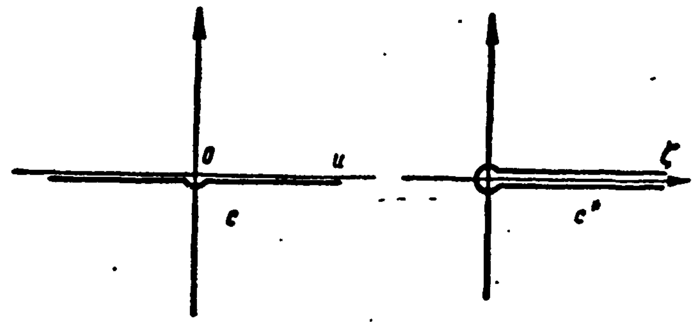


Fig. 54. Contours of Integration, c and c*.

7. Ideal Signal Limitation in the Presence of Noise

As an example to illustrate the contour-integral method, let us examine the correlation function and power spectrum of a process at the output of an ideal limiter with a characteristic of

$$y=f(x)=\begin{cases}a_0, & (x>x_0)\\ 0, & (x<x_0),\end{cases} \tag{7.97}$$

if at its input there acts the sum of a harmonic signal and noise which constitutes a normal, stationary random process, the power spectrum of which is concentrated in a narrow frequency band about the signal-carrier frequency.

It is obvious that the nonlinear characteristic (7.97) is a special case of (7.95) when $\nu = 0$, and therefore the solution to the problem at hand is provided by

formulas (7.90), (7.96) if in them it is assumed that $\nu = 0$.

Let us first consider the case when the signal is absent, i.e., when $a = 0$. Then from (7.96) it follows, that all the coefficients h_{nk} turn to zero, with the exception of the coefficients corresponding to $n = 0$.

Employing the technique used on p. 135, we find

$$h_{0k} = \frac{i^{k-1} a_0}{2\pi} \int_c u^{k-1} e^{-\frac{\sigma^2 u^2}{2}} e^{-i x_0 u} du = \\ = \frac{d^{k-1}}{d x_0^{k-1}} \frac{a_0}{2\pi} \int_c e^{-\frac{\sigma^2 u^2}{2}} e^{-i x_0 u} du \tag{7.98}$$

When $k \geqslant 1$, from (7.98) we find (cf. Appendix VII)

$$h_{0k} = \frac{a_0}{\sqrt{2\pi}} \frac{d^{k-1}}{d x_0^{k-1}} e^{-\frac{x_0^2}{2\sigma^2}} = \frac{a_0}{\sigma^{k-1} \sqrt{2\pi}} H_{k-1}\left(\frac{x_0}{\sigma}\right) e^{-\frac{x_0^2}{2\sigma^2}}. \tag{7.99}$$

To the direct component of the limited noise there corresponds a term equal, when $k = 0$, to

$$h_{0,0} = a_0\left[1 - F\left(\frac{x_0}{\sigma}\right)\right], \tag{7.100}$$

where F is the Laplace function (cf. Section 7, Ch. I).

Substituting (7.99) and (7.100) into (7.92) and taking into account (7.93) - (7.94'), we obtain an expression for the correlation function of normally distributed noise, which has passed through the ideal limiter, in the following form:

$$\begin{gathered} B(\tau) = a_0^2 \sigma^2 \left[1 - F\left(\frac{x_0}{\sigma}\right)\right]^2 + \frac{a_0^2 \sigma^2}{2\pi} e^{-\frac{x_0^2}{\sigma^2}} \left\{ \sum_{k=1}^{\infty} H_{2k-1}^2\left(\frac{x_0}{\sigma}\right) \times \right. \\ \times \frac{\binom{2k}{k}}{(2k)!\,2^{2k}} R_0^{2k}(\tau) + \\ + \left[\sum_{k=1}^{\infty} H_{2k-2}^2\left(\frac{x_0}{\sigma}\right) \frac{\binom{2k-1}{k-1}}{(2k-1)!\,2^{2k-2}} R_0^{2k-1}(\tau)\right] \cos \omega_0 \tau + \\ + \sum_{r=2}^{\infty} \left[\sum_{k=r}^{\infty} H_{2k-2}^2\left(\frac{x_0}{\sigma}\right) \frac{\binom{2k-1}{k-r}}{(2k-1)!\,2^{2k-2}} R_0^{2k-1}(\tau)\right] \cos(2r-1)\omega_0 \tau + \\ \left. + \sum_{r=1}^{\infty} \left[\sum_{k=r}^{\infty} H_{2k-1}^2\left(\frac{x_0}{\sigma}\right) \frac{\binom{2k}{k-r}}{(2k)!\,2^{2k-1}} R_0^{2k}(\tau)\right] \cos 2r\omega_0 \tau \right\}. \end{gathered} \tag{7.101}$$

From this expression there can, by means of a Fourier transformation, be determined the power spectrum of limited noise. This spectrum has a form characteristic to nonlinear transformations, i.e., (besides the direct component) it consists of a video band and of bands situated about frequency ω_0 and about harmonics of that frequency. The energy distribution between the video band and the carrier-frequency harmonics depends on the limiting level.

Let us make use of (7.101) in order to examine in somewhat greater detail the case of low limiting, in which the mean-square value of the noise amplitude is much greater than the height of the limiting level $x_0 \ll \sigma$. Since when k is odd (7.100) contains only even powers of x_0/σ, and when k is even only such odd powers, therefore ignoring the powers of x_0/σ higher than the first, we obtain for the case at hand

$$
\begin{aligned}
B(\tau) = {} & \frac{a_0^2 c^2}{4}\left(1 - \frac{4}{\sqrt{2\pi}}\frac{x_0}{\sigma}\right) + \frac{a_0^2 c^2}{2\pi}\left\{\left[R_0(\tau) + \right.\right. \\
& \left. + \sum_{n=2}^{\infty} \frac{[(2n-3)!!]^2 \binom{2n-1}{n-1}}{(2n-1)!\,2^{2n-2}} R_0^{2n-1}(\tau)\right] \cos \omega_0 \tau + \\
& \left. + \sum_{r=2}^{\infty} \left[\sum_{n=r}^{\infty} \frac{[(2n-3)!!]^2 \binom{2n-1}{n-r}}{(2n-1)!\,2^{2n-2}} R_0^{2n-1}(\tau)\right] \cos (2r-1)\,\omega_0 \tau \right\} + \\
& + \frac{a_0^2 c^2 x_0}{2\pi\sigma}\left\{\sum_{n=1}^{\infty} \frac{[(2n-1)!!]^2 \binom{2n}{n}}{(2n)!\,2^{2n}} R_0^{2n}(\tau) + \right. \\
& \left. + \sum_{r=1}^{\infty} \left[\sum_{n=r}^{\infty} \frac{[(2n-1)!!]^2 \binom{2n}{n-r}}{(2n)!\,2^{2n-1}} R_0^{2n}(\tau)\right] \cos 2r\omega_0 \tau \right\}.
\end{aligned}
\tag{7.102}
$$

It can be seen from (7.102) that when $x_0 = 0$, the spectrum of the limited noise is concentrated only in the vicinity of the carrier frequency ω_0 and its odd harmonics [cf. the terms in (7.102) enclosed in the parentheses]. When $x_0 \neq 0$, but $x_0 \ll \sigma$, combination spectral components appear in the video band, and in the bands situated about the even harmonics of ω_0, but the power corresponding to these parts of the spectrum is much less than in the bands about the odd harmonics of ω_0.

Figure 55 shows the power spectrum for $x_0 \ll \sigma$ with the condition that the noise spectrum at the limiter input is uniform in band Δ. The spectrum in the

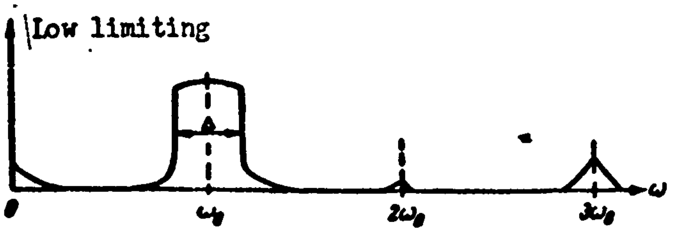


Fig. 55. Noise spectrum with low limiting level.

vicinity of carrier frequency ω_0 is determined by the Fourier transformation of the expression

$$B_1(\tau)=\frac{a_0^2\sigma^2}{2\pi}\left[R_0(\tau)+\sum_{n=2}^{\infty}\frac{[(2n-3)!!]^2\binom{2n-1}{n-1}}{(2n-1)!2^{2n-2}}R_0^{2n-1}(\tau)\right]\cos\omega_0\tau. \qquad (7.103)$$

The first, principal term in (7.103) repeats the form of the input spectrum. The difference of the form of the output spectrum from that of the input spectrum is determined by the succeeding terms, whose influence is, however, insignificant. The form of the spectrum in the vicinity of the carrier frequency is shown in Figure 56. In the same figure the broken line indicates the power spectrum of noise at the limiter input. It can be shown that when $x_0 \longrightarrow 0$ the area of the section of the output spectrum in the vicinity of carrier frequency ω_0, shown in Figure 56, is equal to 0.8 of the area of the input spectrum, i.e., 20% of the noise power becomes the power of the odd harmonics of the carrier.

Let us note, that these same power relationships are preserved with the ideal limiting of a sinusoidal voltage, for which $x_0 = 0$, since in this case the output voltage has the form of a periodic pulse sequence with a density of 0.5.

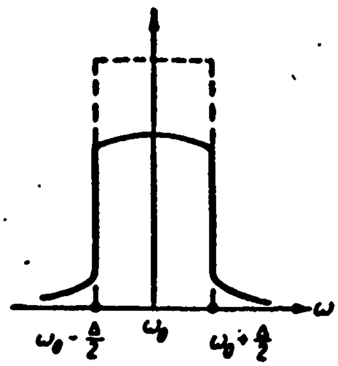


Fig. 56. Spectrum of noise in vicinity of carrier frequency.

With an increase in the limiting level x_0/σ, the distribution of power to various parts of the power spectrum changes substantially. Thus, for instance, with $x_0 = \sigma$ the principal part of the mean power of the process at the output of the limiter is concentrated in the video band and about the frequency ω_0, the power corresponding to the video band amounting to approximately 25% of the total power, and that corresponding to carrier frequency ω_0 amounting to 50% of the total power (Fig. 57). The same phenomenon also takes place in the high limiting of sinusoidal voltage; here the density of the square pulses at the output of the limiter diminishes, and with it also the power of the first harmonic of the cadence frequency.

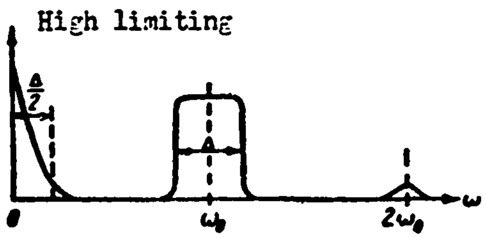


Fig. 57. Noise spectrum with high limiting level.

Let us consider what changes are undergone by the correlation function and the power spectrum if a signal is present, restricting ourselves to the case when the abscissa of the operating point coincides with the cutoff voltage ($x_0 = 0$). Then from (7.96) there follows (cf. Appendix VI)

$$h_{nk} = \frac{i^{n+k-1}a_n}{2\pi}\int_c u^{k-1}J_n(au)\,e^{-\frac{\sigma^2u^2}{2}}du = \frac{a_n}{2n!}\left(\frac{\sqrt{2}}{\sigma}\right)^k\left(\frac{a}{\sigma\sqrt{2}}\right)^n\frac{1}{\Gamma\left(1-\frac{n+k}{2}\right)}\,{}_1F_1\left(\frac{n+k}{2},\, n+1,\, -\frac{a^2}{2\sigma^2}\right). \tag{7.104}$$

Since with a whole negative a gamma-function is limitless, it can be seen from (7.104) that $h_{nk} \equiv 0$ when $n + k = 2r$ ($r = 1, 2, \ldots$). Substituting the expression h_{n0} from (7.104) into the first sum of formula (7.82), we obtain the periodic part of the correlation function

$$B_a(\tau) = \frac{a_0^2}{4} + \frac{a_0^2}{2}\sum_{n=1}^{\infty}\left\{\frac{1}{(2n-1)!}\left(\frac{a}{\sigma\sqrt{2}}\right)^{2n-1}\cdot\frac{1}{\Gamma\left(1-\frac{2n-1}{2}\right)}\times {}_1F_1\left(n-\frac{1}{2},\, 2n,\, -\frac{a^2}{2\sigma^2}\right)\right\}^2\cos(2n-1)\omega_0\tau, \tag{7.105}$$

to which corresponds the discrete part of the power spectrum of the process at the limiter output*:

$$\frac{1}{2\pi}F_2(\omega)=\frac{a_0^2}{4}\delta(\omega)+\frac{a_0^2}{2}\sum_{n=1}^{\infty}\left\{\frac{1}{(2n-1)!}\left(\frac{a}{\sigma\sqrt{2}}\right)^{2n-1}\times\right.$$
$$\left.\times\frac{1}{\Gamma\left(1-\frac{2n-1}{2}\right)}\cdot{}_1F_1\left(n-\frac{1}{2},\ 2n,-\frac{a^2}{2\sigma^2}\right)\right\}^2\delta[\omega-(2n-1)\omega_0]. \tag{7.106}$$

If $\sigma\longrightarrow 0$, then employing the asymptotic resolution of a hypergeometric function (cf. Appendix VI) and taking into account that

$$\Gamma\left(\frac{3}{2}-n\right)\Gamma\left(\frac{1}{2}+n\right)=\frac{2n-1}{2}\cdot(-1)^n\pi,$$

from (7.105') we find

$$\frac{1}{2\pi}F_2(\omega)=\frac{a_0^2}{4}\delta(\omega)+\frac{a_0^2}{2}\cdot\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\delta[\omega-(2n-1)\omega_0],$$

which will provide the power distribution in the spectrum of a periodic sequence of square pulses with a density of $\frac{1}{2}$.

Selecting from the succeeding sums of (7.92) terms for which n = 0, we obtain (deducting the direct component) a correlation function corresponding to the continuous spectrum which is formed by the beats of the noise components. This correlation function coincides with (7.102) if in that expression the last term, which has the magnitude x_0 for a multiplier, is discarded. The remaining terms of (7.92) in these sums ($n>0$) correspond to the continuous spectrum which is formed by the beats of the signal and the noise components. All the even harmonics in (7.92) disappear in the case under consideration.

An explicit expression of the correlation function of a random process at the output of a limiter, in terms of the correlation coefficient of the noise at the input of the limiter and the ratio $\frac{a}{\sigma}$, is obtained after the substitution into (7.92)

* Cf. footnote, p. 275.

of the expressions (7.93) - (7.94') with account being taken of (7.104).*

8. Calculation of Signal/Noise Ratio After Nonlinear Transformation.

The content of many radio-engineering problems lies in evaluating the influence of interference on useful signals. For the solution of problems of this type it is necessary first of all to stipulate a criterion, on the basis of which there can take place a quantitative comparison of the interference-killing features of various systems. The choice of a specific criterion for the evaluation of interference-killing features depends on the method used for separating (observing) the signal at the output of the system.

Very often there is employed as such a criterion the power ratio of signal to noise (or the square root of that ratio), known for short as signal/noise and designated as $\frac{c}{\Pi}$. For calculating this ratio there is employed the theory, set forth above, of the transformations of random-process power spectra in linear and non-linear systems. We emphasize, that the evaluation of interference-killing features by the indicated criterion required a knowledge of only the most general statistical characteristics of random processes, and not the probability distribution of instantaneous values.

Since a unit of radio equipment constitutes a series of standard links, each of which consists of two linear systems with one nonlinear system between them, it is sufficient to consider the manner of calculating the signal/noise ratio at the output of a standard link (Fig. 43).

Let us assume, for the sake of definiteness, that on the input of a standard link there acts a signal, which is a determined function of time, and a noise which

---

* An expression of the correlation function for the case when the process at the limiter input is not a narrow-band one, is cited in [5]. When $x_0 = 0$ and under the condition that the normal process at the input is stationary, the correlation function may be represented in the form of

$$B(\tau) = \frac{a_0^2 c^2}{4}\left[1 + \frac{8}{\pi}\arcsin R(\tau)\right].$$

constitutes a normal stationary random process. Since the signal and the interference pass through the linear system independently, we again have at the input of the nonlinear unit the sum of a determined signal and of normally distributed noise, the spectra of which are deformed according to the frequency characteristic of the linear system, as was indicated in Section 2, Ch. VI. Therefore it is not difficult to compute the power ratio of the signal and the noise at the input of the nonlinear element (abbreviated: signal/noise ratio at input). The power spectrum of the process at the output of the nonlinear unit has a more complex structure. As has been indicated more than once in the present chapter, this spectrum consists of three parts. The discrete part of the spectrum $F_{cXc}(\omega)$ corresponds to the beats between the components of the useful signal. One part of the continuous spectrum, $F_{ш X ш}(\omega)$, is formed by the beats of the noise components, and the other part, $F_{cXш}(\omega)$ is formed by the mutual beats of the signal and noise components.

Let $C(\omega)$ be the frequency characteristic of the linear system (filter) which follows the nonlinear element. Then the power spectrum of the process at the output of a standard link is equal to

$$F(\omega)=C^2(\omega)[F_{c\times c}(\omega)+F_{ш\times ш}(\omega)+F_{c\times ш}(\omega)]. \tag{7.107}$$

In order to compute the signal/noise ratio at the output of a standard link, it is necessary to decide whether the $C^2(\omega)\ F_{cXш}(\omega)$ part of the output spectrum should be imputed to the signal or to the noise. In this connection there result two varieties of power criteria for the evaluation of interference-killing features:

a) the beats between signal components and noise are imputed to noise,

b) the beats between signal components and noise are imputed to the signal.

The first criterion is used in the evaluation of the interference-killing features of communications systems, in which by c/п is understood the square root of the ratio of the power of the useful signal with interference absent in the pass band of the filter, to the power of the interference in the same band, computed with the signal present. In this case the signal/noise ratio is computed according to the

formula

$$\left(\frac{c}{ш}\right)_1^2=\frac{\int\limits_0^\infty F_д(\omega)\,d\omega}{\int\limits_0^\infty F_ш(\omega)\,d\omega}, \qquad (7.108),$$

where

$$F_д(\omega)=C^2(\omega)\,F_{c\times c}(\omega),$$
$$F_ш(\omega)=C^2(\omega)\,[F_{ш\times ш}(\omega)+F_{c\times ш}(\omega)].$$

In problems dealing with the detection of a weak signal, concealed in noise, the presence of beats between the signal and the noise facilitates the detection of the signal, and the employment of the second criterion may prove more expedient. In this case the signal/noise ratio is computed according to the formula

$$\left(\frac{c}{ш}\right)_2^2=\frac{\int\limits_0^\infty F_д(\omega)\,d\omega+\int\limits_0^\infty F_{ш1}(\omega)\,d\omega}{\int\limits_0^\infty F_{ш2}(\omega)\,d\omega}, \qquad (7.109)$$

where

$$F_{ш1}(\omega)=C^2(\omega)\,F_{c\times ш}(\omega),$$
$$F_{ш2}(\omega)=C^2(\omega)\,F_{ш\times ш}(\omega).$$

9. Power Spectrum of Quantization Noise

Along with the quantization of signals by time*, an effective means of augmenting the interference-killing features of radio equipment is the quantization of signals by amplitude. In this case the entire continuous dynamic range is divided into a series of discrete levels. The signal-quantization mechanism at the transmitting end reduces to the transmission, in place of a given instantaneous signal value, of the value of the discrete level closest to it.

The quantization of signals by amplitude makes possible the effective suppression of interference, if only the mean-square value of the interference is small in comparison to the difference between the discrete levels. Quantization leads to signal distortions which are called quantization noise.

Signal quantization by amplitude forms the basis of all pulse-code modulation

---

* cf. footnote, p. 163.

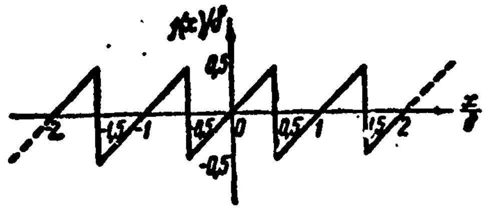


Fig. 58. Non-linear network with sawtooth characteristic.

(P.C.M. [sic. IKM]) systems.

Let us determine the correlation function and power spectrum of quantization noise, assuming that the signal is a normal stationary random process with a zero mean, a dispersion of σ^2 and a correlation coefficient of $R(\tau)$.

The quantization error, i.e., the difference between the initial and the quantized signal, may be regarded as a result of the transformation of an initial signal in the nonlinear system with a saw-tooth characteristic, shown in Figure 58

$$f(x)=x-m\delta, \quad \left(m-\frac{1}{2}\right)\delta<x<\left(m+\frac{1}{2}\right)\delta, \quad m=0, \pm 1, \pm 2, \ldots \tag{7.110}$$

where δ is the distance between the discrete levels.

Employing (7.2) and taking into account (7.110), we find the following expression for the correlation function $B(\tau)$ of quantization noise

$$B(\tau)=\frac{1}{2\pi\sigma^2\sqrt{1-R^2}}\sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty}\int_{\left(m_1-\frac{1}{2}\right)\delta}^{\left(m_1+\frac{1}{2}\right)\delta}\int_{\left(m_2-\frac{1}{2}\right)\delta}^{\left(m_2+\frac{1}{2}\right)\delta}(x_1-m_1\delta)\times (x_2-m_2\delta)e^{-\frac{x_1^2+x_2^2-2Rx_1x_2}{2\sigma^2(1-R^2)}}dx_1dx_2. \tag{7.111}$$

Effecting a replacement of the integration variables

$$x_1-m_1\delta=\frac{\delta}{2}x,$$
$$x_2-m_2\delta=\frac{\delta}{2}y$$

and designating $\beta=\frac{\delta^2}{\sigma^2}$, we obtain

$$B(\tau)=\sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty}\frac{\beta\sigma^2}{32\pi\sqrt{1-R^2}}\int_{-1}^{+1}\int_{-1}^{+1}xy\,e^{-\frac{\beta}{8}\cdot\frac{x^2+y^2-2Rxy}{1-R^2}}\times$$

$$\times e^{-\frac{\beta}{2}\frac{m_1^2+m_2^2+m_1(x-Ry)+m_2(y-Rx)-2Rm_1m_2}{1-R^2}}dxdy,$$

or

$$B(\tau)=\frac{\beta\sigma^2}{32\pi\sqrt{1-R^2}}\int_{-1}^{+1}\int_{-1}^{+1}xyL(x,y)\,e^{-\frac{\beta}{8}\cdot\frac{x^2+y^2-2Rxy}{1-R^2}}dxdy, \tag{7.112}$$

where

$$L(x,y)=\sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty}e^{-\frac{\beta}{2}\cdot\frac{m_1^2+m_2^2+m_1(x-Ry)+m_2(y-Rx)-2Rm_1m_2}{1-R^2}}. \tag{7.113}$$

The summation indices in the expression of function L(x,y) may be separated by effecting the replacement

$$m_1+m_2=n_1,\quad m_1-m_2=n_2. \tag{7.114}$$

The new indices n_1 and n_2 also vary from $-\infty$ to ∞, but they must be either both even, or both odd. After substituting (7.114) into (7.113) and after the simplest of transformations, we have

$$L(x,y)=\sum_{n_1=-\infty}^{\infty}e^{-\frac{\beta}{4}\cdot\frac{2n_1(x+y)+4n_1^2}{1+R}}\sum_{n_2=-\infty}^{\infty}e^{-\frac{\beta}{4}\cdot\frac{2n_2(x-y)+4n_2^2}{1-R}}+$$

$$+\sum_{n_1=-\infty}^{\infty}e^{-\frac{\beta}{4}\frac{(2n_1+1)(x+y)+(2n_1+1)^2}{1+R}}\sum_{n_2=-\infty}^{\infty}e^{-\frac{\beta}{4}\frac{(2n_2+1)(x-y)+(2n_2+1)^2}{1-R}}.$$

The integration variables in integral (7.112) may now also be separated, for which it is sufficient to effect the replacement

$$x=u+v,\quad y=u-v. \tag{7.115}$$

The transformation jacobian of (7.115) is equal to two, and the area of integration in the plane (u,v) is a rhombus bounded by the line u $\pm$ v = $\pm$ 1. In virtue of the full symmetry, integration may take place only with respect to positive values of

variables u and v (one-fourth of the rhombus), the amount of the integral being multiplied by four. In this manner we obtain

$$B(\tau)=\frac{4\beta^2\sigma^2}{16\pi\sqrt{1-R^2}}\int_0^1\int_0^{1-u}(u^2-v^2)e^{-\frac{\beta}{4}\left(\frac{u^2}{1+R}+\frac{v^2}{1-R}\right)}\times$$
$$\times\left\{\sum_{n_1=-\infty}^{\infty}e^{-\frac{2n_1\beta(2u+2n_1)}{4(1+R)}}\sum_{n_2=-\infty}^{\infty}e^{-\frac{2n_2\beta(2v+2n_2)}{4(1-R)}}+\right.\tag{7.116}$$
$$\left.+\sum_{n_1=-\infty}^{\infty}e^{-\frac{(2n_1+1)\beta(2u+2n_1+1)}{4(1+R)}}\sum_{n_2=-\infty}^{\infty}e^{-\frac{(2n_2+1)\beta(2v+2n_2+1)}{4(1-R)}}\right\}dudv.$$

Additional simplification of expression (7.116) may be attained, if the sums entering into it are transformed by means of the Poisson formula (cf., e.g. B. Van der Pol' and Kh. Bremmer. Operatsionoye ischisleniye na osnove dvukhstoronnego preobrazovaniye Laplasa (i.e., Balth Van der Pol and H. Bremmer, "Operational Calculus Based on the Two-sided Laplace Integral" [N. Y., 1950]), For. Lit. Pub. Hse., 1952, p. 131):

$$\sum_{n=-\infty}^{\infty}h(n)=\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}h(x)e^{2\pi inx}dx.$$

Employing this formula, we find

$$\sum_{n_1=-\infty}^{\infty}e^{-\frac{n_1\beta(u+n_1)}{1+R}}=\sqrt{\frac{\pi(1+R)}{\beta}}e^{\frac{\beta}{4}\frac{u^2}{1+R}}\times$$
$$\times\left[1+2\sum_{r=1}^{\infty}e^{-\frac{r^2\pi^2(1+R)}{2\beta}}\cos\frac{r\pi u}{2}\right].\tag{7.117}$$

Analogously

$$\sum_{n_2=-\infty}^{\infty}e^{-\frac{n_2\beta(v+n_2)}{1-R}}=\sqrt{\frac{\pi(1-R)}{\beta}}e^{\frac{\beta}{4}\frac{v^2}{1-R}}\times$$
$$\times\left[1+2\sum_{r=1}^{\infty}e^{-\frac{r^2\pi^2(1-R)}{2\beta}}\cos\frac{r\pi v}{2}\right].\tag{7.118}$$

$$\sum_{n_1=-\infty}^{\infty}e^{-\frac{(2n_1+1)\beta(2u+2n_1+1)}{4(1+R)}}=\sqrt{\frac{\pi(1+R)}{\beta}}e^{\frac{\beta}{4}\frac{u^2}{1+R}}\times$$
$$\times\left[1+2\sum_{r=1}^{\infty}(-1)^re^{-\frac{r^2\pi^2(1+R)}{2\beta}}\cos\frac{r\pi u}{2}\right],\tag{7.119}$$

$$\sum_{n_2=-\infty}^{\infty} e^{-\frac{(2n_2+1)\beta(2v+2n_2+1)}{4(1-R)}} = \sqrt{\frac{\pi(1-R)}{\beta}}\, e^{\frac{\beta}{4}\frac{v^2}{1-R}} \times$$

$$\times\left[1+2\sum_{r=1}^{\infty}(-1)^r e^{-\frac{r^2\pi^2(1-R)}{2\beta}}\cos\frac{r\pi u}{2}\right]. \tag{7.120}$$

Substituting (7.117) - (7.120) into (7.116), and noting that the exponential factors in these sums cancel out with the exponential factor of the integrand function, after elementary transformations we obtain

$$B(\tau)=\frac{\beta c^2}{4}\int_0^1\int_0^{1-u}(u^2-v^2)\left[f_1(1+R,u)\,f_1(1-R,v)+\right. \tag{7.121}$$

$$\left.+f_2(1+R,u)\,f_2(1-R,v)\right]du\,dv,$$

where is designated

$$f_1(a,z)=1+2\sum_{r=1}^{\infty}e^{-\frac{r^2\pi^2 a}{2\beta}}\cos\frac{r\pi z}{2}, \tag{7.122}$$

$$f_2(a,z)=1+2\sum_{r=1}^{\infty}(-1)^r e^{-\frac{r^2\pi^2 a}{2\beta}}\cos\frac{r\pi z}{2}. \tag{7.123}$$

Now the integration in (7.121) may be brought to a conclusion with no particular difficulties, since the integrals subject to computation are tabular. Omitting the algebraic transformations, we cite the final result

$$B(\tau)=\frac{\beta c^2}{\pi^2}\left\{\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-\frac{4n^2\pi^2}{\beta}}\operatorname{sh}\frac{4n^2\pi^2 R}{\beta}+\right.$$

$$+\sum_{\substack{n_1=1\\ n_1\neq n_2}}^{\infty}\sum_{n_2=1}^{\infty}\frac{1}{n_1^2-n_2^2}e^{-\frac{4(n_1^2+n_2^2)\pi^2}{\beta}}\operatorname{sh}\frac{4(n_1^2-n_2^2)\pi^2 R}{\beta}-$$

$$-\sum_{\substack{n_1=1\\ n_1\neq n_2}}^{\infty}\sum_{n_2=1}^{\infty}\frac{1}{\left(n_1-\frac{1}{2}\right)^2-\left(n_2-\frac{1}{2}\right)^2}e^{-\frac{4\left[\left(n_1-\frac{1}{2}\right)^2+\left(n_2-\frac{1}{2}\right)^2\right]\pi^2}{\beta}}\times \tag{7.124}$$

$$\left.\times\operatorname{sh}\frac{4\left[\left(n_1-\frac{1}{2}\right)^2-\left(n_2-\frac{1}{2}\right)^2\right]\pi^2 R}{\beta}\right\}.$$

Let us assume that the difference δ between the discrete levels is much less than the mean-square value σ of the signal. This assumption is almost always realized. Then $\beta \ll 1$. Taking this last inequality into account, it is possible

in (7.124) to neglect the double sums compared with the first sum, and, replacing the hyperbolic sine by its asymptotic expansion, we obtain the following approximate expression for the correlation function of quantization noise, sufficient for most problems of practical interest

$$B(\tau)=\frac{\beta\sigma^2}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-\frac{4n^2\pi^2(1-R)}{\beta}}. \qquad (7.125)$$

The full power of the quantization noise (the dispersion of quantization error) is equal to

$$B(0)=\frac{\beta\sigma^2}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\beta\sigma^2}{2\pi^2}\cdot\frac{\pi^2}{6}=\frac{\delta^2}{12}, \qquad (7.126)$$

i.e., one-twelfth of the square of the distance between the discrete levels. It is not difficult to notice that, in the case under consideration, the dispersion of error coincides with the dispersion of a random variable uniformly distributed over the interval of from 0 to δ. This is so because with a small difference between the discrete levels, the quantization error is sufficiently closely approximated by segments of straight lines (with the exception of those cases, when the signal between the discrete levels passes through the extreme).

Let us determine the power spectrum $F(\omega)$ of the quantization noise with the assumption that the spectrum of the initial signal is uniform in band Δ. The correlation coefficient of such a signal, is in accordance with (6.18), equal to $R(\tau)=\frac{\sin\tau\Delta}{\tau\Delta}$. Then from (7.125), employing Khinchin's theorem, we find

$$F(\omega)=4\frac{\beta\sigma^2}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^{\infty}e^{-\frac{4n^2\pi^2}{\beta}\left(1-\frac{\sin\tau\Delta}{\tau\Delta}\right)}\cos\omega\tau d\tau. \qquad (7.127)$$

Expanding $\frac{\sin\tau\Delta}{\tau\Delta}$ into a series and restricting ourselves to the first two terms (which is permissible, since the integrand functions in (7.127) diminish rapidly with an increase in the magnitude of $\tau\Delta$), we obtain

$$F(\omega)=\frac{2\beta\sigma^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^{\infty}e^{-\frac{2n^2\pi^2}{3\beta}\tau^2\Delta^2}\cos\omega\tau\,d\tau. \tag{7.128}$$

The integrals obtained under the summation sign have already been encountered above (cf. p. 228). Substituting in (7.128) the magnitudes of these integrals, we find the power spectrum of quantization noise in the form of

$$F(\omega)=\frac{\beta\sigma^2}{\pi^2\Delta}\sqrt{\frac{3\beta}{2\pi}}\sum_{n=1}^{\infty}\frac{1}{n^3}e^{-\frac{3\beta\omega^2}{8n^2\pi^2\Delta^2}}. \tag{7.129}$$

The correlation time, in accordance with (5.32) and (5.47) is

$$\tau_0=\frac{F(0)}{4B(0)}=\frac{3}{\pi^2\Delta}\sqrt{\frac{3\beta}{2\pi}}\sum_{n=1}^{\infty}\frac{1}{n^3}=\frac{3\pi}{25,8}\sqrt{\frac{3\beta}{2\pi}}\cdot\frac{1}{\Delta},$$

or

$$\tau_0\approx\frac{\sqrt{\beta}}{4\Delta}=\frac{\delta}{4\sigma\Delta}, \tag{7.130}$$

i.e., the correlation time of the quantization noise is approximately $2/\sqrt{\beta}$ times less than that of the initial signal. When $\beta \ll 1$, correlation between the quantization errors in the consecutive selections of signal values is practically absent. Accordingly, with a reduction of the distance between the discrete levels the power spectrum of the quantization noise becomes uniform in a wider frequency range, with a simultaneous decrease in the maximum of spectrum density.

Literature

1. V. I. Bunimovich. Fluktuatsionnyye protsessy v radiopriyemnykh ustroystvakh (Fluctuation Processes in Radio Receivers). "Sovetskoye Radio" ("Soviet Radio") Publishing House, 1951.

2. S. O. Rice. Mathematical Analysis of Random Noise. BSTJ, 23, No. 3, July 1944; 24, No. 1, Jan. 1945. (A translation [into Russian] is available in the symposium "Teoriya peredachi elektrisheskikh signalov pri nalichii pomekh" (Theory of the Transmission of Electrical Signals in the Presence of Noise), For. Lit. Pub. Hse.,

1953).

3. W. E. Thompson. The Response of a Non-Linear System to Random Noise. Pros (sic) IEE, 1955, p. 102, No. 1.

4. W. B. Davenport. Signal-to-noise Ratios in Band-Pass Limiters. Journ. Appl. Phys., 24, No. 6, June 1953.

5. I. N. Amiantov, V. I. Tikhonov. Vozdeystviye normal'nykh fluktuatsiy na tipovyye nelineynyye elementy (The Action of Normal Fluctuations on Standard Non-linear Elements). Izvestiya AN SSSR, OTN (Otdel Tekhnicheskikh Nauk - Technical Sciences Division) No. 4, 1956.

6. W. R. Bennett. Spectra of Quantized Signals. Bell System Techn. Jour., 27, No. 3, 1948.

7. G. A. Levin, M. M. Golovchinev. Analiz shumov kvantovaniya pri impul'snokodovoy modulatsii (The Analysis of Quantization Noise in Pulse-Code Modulation). "Radio-technika" ("Radio Engineering"), No. 8, 1955.

8. L. Campbell. Rectification of Two Signals in Random Noise. Trans. IRE, IT-2, 1956, No. 3.

Chapter VIII

STATISTICAL CHARACTERISTICS OF THE ENVELOPE AND PHASE OF A NORMAL RANDOM PROCESS

1. Formulation of the Problem, and its General Solution.

Let us examine a narrow-band linear system, at the input of which there acts a determined process (the signal) together with a stationary normal random process (noise).

In accordance with the results of Sect. 5, Ch. VI the stationary process at the output of a narrow-band linear system with a resonance frequency of ω_0 may be represented in the form of

$$A(t)\cos\omega_0 t + C(t)\sin\omega_0 t,$$

where A(t) and C(t) are independent, stationary random functions which have normal laws of distribution with zero mean values, dispersions of σ^2 and correlation coefficients of $R_0(\tau)$.

Let the signal S(t), which has passed through a linear system, consist of a high-frequency vibration of the frequency ω_0, modulated by amplitude and by phase, i.e.,

$$S(t) = u(t)\cos\omega_0 t + v(t)\sin\omega_0 t.$$

Then the random process of the linear system under examination is described by the random function

$$\xi(t) = [(A(t) + u(t)]\cos\omega_0 t + [C(t) + v(t)]\sin\omega_0 t, \tag{8.1}$$

which may be represented in the form of

$$\xi(t) = E(t)\cos[\omega_0 t + \varphi(t)], \tag{8.1'}$$

where E(t) and φ (t) are the envelope and phase* of the random process ξ(t), defined

* In some works (cf. e.g., [1]) by the phase of a process is meant the sum $\omega_0(t) + \varphi(t)$. In avoidance of error it should be kept in mind, that here by phase is meant only the random function φ (t).

by the formulas

$$E(t)=\sqrt{[A(t)+u(t)]^2+[C(t)+v(t)]^2}, \tag{8.2}$$

$$\varphi(t)=-\operatorname{arctg}\frac{C(t)+v(t)}{A(t)+u(t)}. \tag{8.2'}$$

In many applications considerable interest is afforded by the problem of determining the distribution functions of the envelope E(t) and phase φ(t) of the normal random process (8.1), a general solution of which is indicated in Sect. 7, Ch. VI. Formulas (6.79) and (6.80), there cited, provide a basic solution to the indicated problem. For this it is sufficient, to insert the explicit expressions of n-dimensional distribution functions in place of w_{n1} and w_{n2}. The practical application of these formulas is complicated by a considerable difficulty in the computation of the participating integrals. We shall limit ourselves here to obtaining the distribution functions of the first two orders.

2. Distribution Functions of the Envelope.

The problem of obtaining the first distribution function of the envelope of a normal distribution function - (8.1) coincides fully with the problem solved in Sect. 5, Chapter III of the probability density of the length of a plane vector, the components of which are independent and are normally distributed with parameters of $[u(t), \sigma]$ and $[v(t), \sigma]$, where σ^2 is the dispersion of the stationary part of process $\xi(t)$. Employing (3.38), we write the first distribution function of the envelope

$$W_1(r,t)=\frac{r}{\sigma^2}e^{-\frac{r^2+a^2(t)}{2\sigma^2}}I_0\left[\frac{ra(t)}{\sigma^2}\right], \quad r>0 \tag{8.3}$$

$$W_1(r,t)\equiv 0, \quad r<0,$$

where

$$a^2(t)=u^2(t)+v^2(t).$$

Thus, the first distribution function of the envelope of a normal process coincides in the general case with the generalized Rayleigh law of distribution. This

function for various fixed values* of α/σ is shown in Figure 24. As the ratio α/σ is increased, the distribution law of the envelope approaches the normal [cf. (3.40)].

When the signal is absent ($\alpha = 0$), the distribution law (8.3) turns into an ordinary Rayleigh distribution law (corresponding in the indicated figure to the first curve on the left)

$$W_1(r) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}, \quad r > 0,$$
$$W_1(r) \equiv 0, \quad r < 0. \tag{8.3'}$$

The integral distribution law of the envelope, i.e., the probability that E(t) does not exceed a given magnitude r, follows directly from (8.3)

$$P\{E < r\} = \frac{1}{\sigma^2} \int_0^r r e^{-\frac{r^2 + a^2(t)}{2\sigma^2}} I_0\left(\frac{r a(t)}{\sigma^2}\right) dr.$$

Integrating by parts and employing the relationship

$$\int u^n I_n(au)\, du = \frac{u^n}{a} I_n(au),$$

we obtain

$$P\{E < r\} = e^{-\frac{r^2 + a^2}{2\sigma^2}} \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n I_n\left(\frac{ar}{\sigma^2}\right). \tag{8.4}$$

The curves of the integral distribution law (8.4) were shown in Figure 25.

Passing to the determination of the second distribution function of the envelope of a normal process, we shall restrict our detailed computations to the case where the signal is absent.

Employing (5.94) and (6.79) when n = 2, we obtain the desired two-dimensional distribution function of the envelope

$$W_2(r_1, r_2 \tau) = \frac{r_1 r_2}{(2\pi\sigma^2)^2 (1 - R_0^2)} \int_0^{2\pi}\int_0^{2\pi} e^{-\frac{r_1^2 - 2R_0 r_1 r_2 \cos(\theta_2 - \theta_1) + r_2^2}{2\sigma^2(1 - R_0^2)}} d\Theta_1 d\Theta_2 =$$
$$= \frac{r_1 r_2}{(2\pi\sigma^2)^2 (1 - R_0^2)} e^{-\frac{r_1^2 + r_2^2}{2\sigma^2(1 - R_0^2)}} \int_0^{2\pi}\int_0^{2\pi} e^{\frac{R_0 r_1 r_2 \cos(\theta_2 - \theta_1)}{\sigma^2(1 - R_0^2)}} d\Theta_1 d\Theta_2, \tag{8.5}$$
$$r_1 > 0, \quad r_2 > 0.$$

---

* I.e., for the fixed instant of time t = t*; thereafter α(t) should always be considered in a fixed instant of time.

For computation of the double integral in the right part of (8.5) it is useful to effect a replacement of the variables

$$u=\frac{\theta_1-\theta_2}{\sqrt{2}}, \quad v=\frac{\theta_1+\theta_2}{\sqrt{2}}, \tag{8.6}$$

which corresponds to a rotation of the coordinate axes through an angle of 45° (Figure 59). It is not difficult to satisfy oneself that the transformation jacobian of (8.6) is equal to unity.

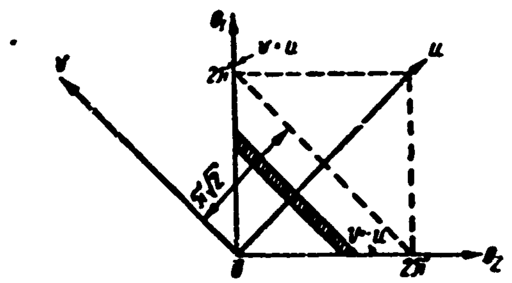


Fig. 59. Rotation of the coordinate axes.

First integrating along v (Figure 59), we find

$$\begin{aligned}
&\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} e^{\frac{R_0 r_1 r_2 \cos(\theta_1-\theta_2)}{\sigma^2(1-R_0^2)}}\, d\theta_1 d\theta_2 = \\
&= \frac{2}{(2\pi)^2}\int_0^{\pi\sqrt{2}}\int_{-u}^{u} e^{\frac{R_0 r_1 r_2 \cos u\sqrt{2}}{\sigma^2(1-R_0^2)}}\, dv\, du = \\
&= \frac{2}{(2\pi)^2}\int_{-\pi}^{\pi} (z+\pi)\, e^{-\frac{R_0 r_1 r_2}{\sigma^2(1-R_0^2)}\cos z}\, dz. \\
&= \frac{4}{(2\pi)^2}\int_0^{\pi\sqrt{2}} u\, e^{\frac{R_0 r_1 r_2 \cos u\sqrt{2}}{\sigma^2(1-R_0^2)}}\, du = \frac{2}{(2\pi)^2}\int_0^{2\pi} y\, e^{\frac{R_0 r_1 r_2 \cos y}{\sigma^2(1-R_0^2)}}\, dy =
\end{aligned} \tag{8.7}$$

The replacement of $y = z+\pi$ resulted in two integrals, one of which, by virtue of the oddness of the integrand function, is

$$\int_{-\pi}^{\pi} z\, e^{-\frac{R_0 r_1 r_2}{\sigma^2(1-R_0^2)}\cos z}\, dz = 0,$$

and the other (cf. Section 5, Chapter III) is equal to

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\frac{R_0 r_1 r_2}{\sigma^2(1-R_0^2)}\cos z}\, dz = I_0\left[-\frac{R_0 r_1 r_2}{\sigma^2(1-R_0^2)}\right].$$

Thus we obtain the two-dimensional distribution function of the envelope of a stationary normal random process

$$W_2(r_1, r_2, \tau)=\frac{r_1 r_2}{\sigma^4(1-R_0^2)} e^{-\frac{r_1^2+r_2^2}{2\sigma^2(1-R_0^2)}} I_0\left[\frac{R_0 r_1 r_2}{\sigma^2(1-R_0^2)}\right], \; r_1>0, \; r_2>0, \qquad (8.8)$$
$$W_2(r_1, r_2, \tau)\equiv 0, \quad r_1<0, \; r_2<0.$$

When $\tau \longrightarrow \infty$, $R_o \longrightarrow 0$, and from (8.8) there follows

$$W_2(r_1, r_2, \infty)=\frac{r_1}{\sigma^2} e^{-\frac{r_1^2}{2\sigma^2}} \cdot \frac{r_2}{\sigma^2} e^{-\frac{r_2^2}{2\sigma^2}},$$

i.e., the two-dimensional distribution function is, as was to be expected, equal to the product of the ordinary Rayleigh distribution functions which represent the one-dimensional probability densities of the envelope of a stationary normal random process for $\tau \rightarrow \infty$.

The general expression for the n-dimensional distribution function of the envelope of a stationary normal process has been obtained in [9] in the form of a product of exponential and Bessel functions.

The case when a regular signal is present is considered analogously, although the computations turn out to be more cumbersome. Here there will be cited only the final expression of the two-dimensional distribution function of the envelope

$$W_2(r_1, r_2, \tau, t)=\frac{r_1 r_2}{\sigma^4(1-R_0^2)} e^{-\frac{r_1^2+r_2^2+a_1^2+a_2^2-2a_1 a_2 R_0}{2\sigma^2(1-R_0^2)}} \times$$
$$\times\sum_{m=0}^{\infty}\varepsilon_m I_m\left[\frac{R_0 r_1 r_2}{\sigma^2(1-R_0^2)}\right] I_m\left[\frac{a_1-R_0 a_2}{\sigma^2(1-R_0^2)} r_1\right] I_m\left[\frac{a_2-R_0 a_1}{\sigma^2(1-R_0^2)} r_2\right], \qquad (8.9)$$

where $\alpha_1 = \alpha(t)$, $\alpha_2 = \alpha(t+\tau)$, and $\varepsilon_0 = 1$, $\varepsilon_m = 2$ with $m>0$.

If the signal constitutes a harmonic vibration of the frequency ω_0 and the amplitude u_0, then $\alpha_1 = \alpha_2 = u_0$, and from (5.9) there follows

$$W_2(r_1, r_2, \tau)=\frac{r_1 r_2}{\sigma^4(1-R_0^2)} e^{-\frac{r_1^2+r_2^2}{2\sigma^2(1-R_0^2)}} e^{\frac{u_0^2}{\sigma^2(1+R_0)}} \times$$
$$\times\sum_{m=0}^{\infty}\varepsilon_m I_m\left[\frac{R_0 r_1 r_2}{\sigma^2(1-R_0^2)}\right] I_m\left[\frac{u_0 r_1}{\sigma^2(1+R_0)}\right] I_m\left[\frac{u_0 r_2}{\sigma^2(1+R_0)}\right]. \qquad (8.9')$$

When $\tau \longrightarrow \infty$, $R_o \longrightarrow 0$ and from (8.9) we find

$$W_2(r_1, r_2, t)=\frac{r_1}{\sigma^2} e^{-\frac{r_1^2+a_1^2}{2\sigma^2}} I_0\left(\frac{r_1 a_1}{\sigma^2}\right)\cdot\frac{r_2}{\sigma^2} e^{-\frac{r_2^2+a_2^2}{2\sigma^2}} I_0\left(\frac{r_2 a_2}{\sigma^2}\right),$$

i.e., the two-dimensional distribution function is equal to the product of the one-dimensional distribution functions (8.3).

3. Correlation Function of the Envelope

Having the expressions of the two-dimensional probability density of the envelope, it is possible to find its correlation function, since the latter is a second mixed moment of distribution. To compute in this connection the double integral, it is expedient to employ the method set forth in Sect. 6, Ch. VI, of expanding the two-dimensional probability density into a series of the respective orthogonal polynomials.

Let us examine in detail the sequence of determining the correlation function of the envelope of a stationary normal random process. The two-dimensional distribution function is here determined by formula (8.8), and the one-dimensional distribution corresponding to it — by formula (8.3). If the function $\frac{r}{\sigma\sqrt{2}} e^{-\frac{r^2}{2\sigma^2}}$ over the interval $(0, \infty)$ is taken as a weighting function, then to this weighting function there corresponds the aggregate of orthogonal Laguerre* polynomials $L_n^{(1/2)}\left(\frac{r^2}{2\sigma^2}\right)$.

The expansion of two-dimensional distribution function (8.8) into a series of these polynomials has the form of

$$\frac{r_1 r_2}{\sigma^4(1-R_0^2)} e^{-\frac{r_1^2+r_2^2}{2\sigma^2(1-R_0^2)}} I_0\left[\frac{R_0 r_1 r_2}{\sigma^2(1-R_0^2)}\right]=\frac{r_1 r_2}{\sigma^4} e^{-\frac{r_1^2+r_2^2}{2\sigma^2}}\times \times\sum_{n=0}^{\infty}\frac{R_0^{2n}}{(n!)^2} L_n^{(1/2)}\left(\frac{r_1^2}{2\sigma^2}\right) L_n^{(1/2)}\left(\frac{r_2^2}{2\sigma^2}\right). \tag{8.10}$$

Employing expansion (8.10), we represent the correlation function of the envelope by

* By definition, the Laguerre polynomials are equal to

$$L_n^{(\alpha)}(x)=(-1)^n x^{-\alpha} e^x \frac{d^n}{dx^n}(x^{n+\alpha} e^{-x})$$

(cf. V. L. Goncharov "Teoriya interpolirovaniya i priblizheniya funktsiy" (Theory of Interpolation and Approximation of Functions), Moskva, Gostekhizdat, 1954).

the series

$$B_0(\tau)=\int_0^\infty\int_0^\infty r_1r_2w_2(r_1,\ r_2,\ \tau)\,dr_1dr_2=$$

$$=\sum_{n=0}^{\infty}\frac{R_0^{2n}}{(n!)^2}\cdot\frac{1}{\sigma^4}\int_0^\infty\int_0^\infty r_1^2\,r_2^2\,L_n^{(1/2)}\left(\frac{r_1^2}{2\sigma^2}\right)L_n^{(1/2)}\left(\frac{r_2^2}{2\sigma^2}\right)e^{-\frac{r_1^2+r_2^2}{2\sigma^2}}dr_1dr_2,$$

and since the variables in the double integral are separable,

$$B_0(\tau)=\sum_{n=0}^{\infty}\frac{R_0^{2n}}{(n!)^2}\left[\frac{1}{\sigma^2}\int_0^\infty r^2L_n^{(1/2)}\left(\frac{r^2}{2\sigma^2}\right)e^{-\frac{r^2}{2\sigma^2}}dr\right]^2, \tag{8.11}$$

or

$$B_0(\tau)=\sigma^2\sum_{n=0}^{\infty}R_0^{2n}(\tau)\,c_n^2, \tag{8.12}$$

where

$$c_n=\frac{1}{n!}\int_0^\infty x^2L_n^{(1/2)}\left(\frac{x^2}{2}\right)e^{-\frac{x^2}{2}}dx. \tag{8.12'}$$

The computation of integral (8.12') is carried out very simply, if use is made of the link which exists between the Laguerre and the Hermite polynomials (cf. V. I. Smirnov, Kurs vysshey matematiki (Course in Higher Mathematics), V. III, part II, GITTI, 1949, p. 576):

$$L_n^{(1/2)}\left(\frac{x^2}{2}\right)=(-1)^n\frac{1}{2^{n+\frac{1}{2}}}H_{2n+1}(x). \tag{8.13}$$

Then

$$c_n=\frac{(-1)^n}{2^{n+\frac{1}{2}}n!}\int_0^\infty x^2H_{2n+1}(x)\,e^{-\frac{x^2}{2}}dx=$$

$$=\frac{(-1)^{n+1}}{2^{n+\frac{1}{2}}n!}\int_0^\infty x^2\,\frac{d^{2n+1}}{dx^{2n+1}}e^{-\frac{x^2}{2}}dx$$

and integrating by parts twice, we find*

$$c_n=\frac{(-1)^{n+1}\sqrt{2}}{2^n n!}H_{2n-2}(0)=\frac{(2n-3)!!\sqrt{2}}{2^n n!},\quad n\geqslant 2. \tag{8.14}$$

For n = 0 and n = 1 it follows directly from (8.12') and (8.13) that*

$$c_0=\sqrt{2}\int_0^\infty\frac{x^2}{2}e^{-\frac{x^2}{2}}dx=\sqrt{2}\int_0^\infty ye^{-y}dy=\sqrt{2},$$

$$c_1=-\frac{\sqrt{2}}{4}\int_0^\infty x^2(x^3-3x)\,e^{-\frac{x^2}{2}}dx=\frac{3\sqrt{2}}{2}- \tag{8.14'}$$

* cf. Appendix VII.

$$-\frac{\sqrt{2}}{4}\int_0^\infty x^5 e^{-\frac{x^2}{2}}dx=\frac{\sqrt{2}}{2}. \tag{8.14''}$$

Substituting (8.14) - (8.14") into (8.12), we obtain the expression for the correlation function of the envelope of a stationary normal random process

$$B_0(\tau)=2\sigma^2\left(1+\frac{R_0^2}{4}+\sum_{n=2}^{\infty}\frac{[(2n-3)!!]^2}{2^{2n}(n!)^2}R_0^{2n}\right), \tag{8.15}$$

which coincides (with an accuracy to the last constant factor) with (3.74)*. This correspondence is to have been expected, since the power spectrum of the envelope is the low-frequency part of the spectrum of a stationary normal random process after its linear detection.

For the case of a harmonic signal present with an amplitude of u_0, the correlation function of the envelope has been computed in [2] and has the following form:

$$\begin{aligned} B_0(\tau)=2\sigma^2(1-R_0^2)^2 e^{-\frac{u_0^2}{2\sigma^2}}\sum_{m=0}^{\infty}\frac{(2m)!\,[(2m+1)!!]^2}{2^{2m}(m!)^2}\times \\ \times\sum_{n=0}^{2m}\frac{R_0^{2m-n}}{(2m-n)!(n!)^2}\left(\frac{u_0^2}{2\sigma^2}\cdot\frac{1-R_0}{1+R_0}\right)^n\times \\ \times F_1\left(n-2m-2,\ n+1,\ -\frac{1-R_0}{1+R_0}\cdot\frac{u_0^2}{2\sigma^2}\right). \end{aligned} \tag{8.15}$$ [si

4. Nonlinear Transformations of the Envelope. The Square-law Detector.

It has been noted in Sect. 7, Ch.VI, that the random process, obtained by the nonlinear transformation of a narrow-band random process, may be represented by the series

$$\begin{aligned}\eta(t)=f_0(E)+f_1(E)\cos[\omega_0 t-\varphi(t)]+ \\ +f_2(E)\cos[2\omega_0 t-2\varphi(t)]+\ldots,\end{aligned}$$

where

$$f_n(E)=\frac{\varepsilon_n}{\pi}\int_0^\pi f(E\cos\psi)\cos n\psi\,d\psi,\quad (\varepsilon_0=1,\ \varepsilon_n=2,\ n>0).$$

Each of the components of random process $\eta(t)$ is of the same nature as random process $\xi(t)$, i.e., represents a product of the slowly changing envelope $f_n(E)$ by

* The indicated correspondence becomes obvious, if under the summation sign, $\frac{\binom{2n}{n}}{(2n)!}$ is replaced by $\frac{1}{(n!)^2}$.

$\cos n\,[\omega_0 t - \varphi(t)]$, the power spectrum of $f_n(E)$ being principally concentrated in the frequency range close to $n\omega_0$. In particular, the spectrum of $f_0(E)$, as well as the spectrum of E, is concentrated in the range of low frequencies adjacent to $\omega = 0$.

It is of interest to obtain the first two distribution functions of the random process $f_0(E)$, which is obtained as a result of the nonlinear transformation of the envelope of a normal random process. For this it is sufficient to effect the appropriate replacement of variables in (8.3) and (8.9) according to the formulas of Sect. 1, Ch. III.

Let us, for example, consider what will be the distribution functions of the envelope of a narrow-band normal random process after square-law detection. In the case under consideration

$$f_0[E(t)] = E^2(t). \tag{8.16}$$

Let us begin by determining the two-dimensional distribution function of $E^2(t)$. The problem is, in formula (8.9) to make the transition from the variables r_1 and r_2 to the variables

$$\rho_1 = r_1^2, \quad \rho_2 = r_2^2. \tag{8.17}$$

Although the inverse function $r = \pm\sqrt{\rho}$ is two-valued, nevertheless, since $r_1 > 0$, $r_2 > 0$, to each point in the plane (ρ_1, ρ_2) there will correspond only one point in the plane (r_1, r_2). The transformation jacobian of (8.17) is equal to

$$\frac{\partial(\rho_1, \rho_2)}{\partial(r_1, r_2)} = \begin{vmatrix} 2r_1 & 0 \\ 0 & 2r_2 \end{vmatrix} = 4r_1 r_2.$$

Employing (3.10) and taking into account that $\frac{\partial(r_1, r_2)}{\partial(\rho_1, \rho_2)} = \frac{1}{4r_1 r_2}$, we obtain the two-dimensional distribution function of the square of the envelope

$$\begin{gathered} W_2(\rho_1, \rho_2, \tau, t) = \frac{1}{4\sigma^4(1-R_0^2)} e^{-\frac{\rho_1+\rho_2+a_1^2+a_2^2-2a_1a_2R_0}{2\sigma^2(1-R_0^2)}} \times \\ \times \sum_{m=0}^{\infty} \varepsilon_m I_m\left[\frac{R_0\sqrt{\rho_1\rho_2}}{\sigma^2(1-R_0^2)}\right] I_m\left[\frac{a_1 - R_0 a_2}{\sigma^2(1-R_0^2)}\sqrt{\rho_1}\right] \times \\ \times I_m\left[\frac{a_2 - R_0 a_1}{\sigma^2(1-R_0^2)}\sqrt{\rho_2}\right], \\ \rho_1 > 0, \quad \rho_2 > 0. \end{gathered} \tag{8.18}$$

If the signal is absent, then $\alpha_1 = \alpha_2 \equiv 0$, and

$$W_2(\rho_1, \rho_2, \tau) = \frac{1}{4\sigma^4(1-R_0^2)} e^{-\frac{\rho_1+\rho_2}{2\sigma^2(1-R_0^2)}} I_0\left[\frac{R_0\sqrt{\rho_1\rho_2}}{\sigma^2(1-R_0^2)}\right]. \tag{8.19}$$

The one-dimensional distribution function is not difficult to obtain from (8.18), if $\tau \to 0$. Then

$$W_1(\rho, t) = \frac{1}{2\sigma^2} e^{-\frac{\rho+\alpha^2}{2\sigma^2}} I_0\left(\frac{\alpha\sqrt{\rho}}{\sigma^2}\right), \ \rho > 0. \tag{8.20}$$

When $\alpha \gg \sigma$, the Bessel function may be replaced by its asymptotic approximation

$$I_0\left(\frac{\alpha\sqrt{\rho}}{\sigma^2}\right) \sim \sqrt{\frac{\sigma^2}{2\pi\alpha\sqrt{\rho}}} e^{\frac{\alpha\sqrt{\rho}}{\sigma^2}}.$$

Then the distribution law of (8.20) may be represented in the form (cf. #5, Chapter III) of

$$W_1(\rho, t) = \frac{1}{2\rho^{\frac{1}{4}}\sqrt{2\pi\alpha\sigma^2}} e^{-\frac{(\sqrt{\rho}-\alpha)^2}{2\sigma^2}}, \ \rho > 0. \tag{8.21}$$

A curve of the distribution function (8 ?) for several fixed values of $\frac{\alpha}{\sigma}$ is shown in Figure 60. Curve 1 corresponds to the absence of a signal.

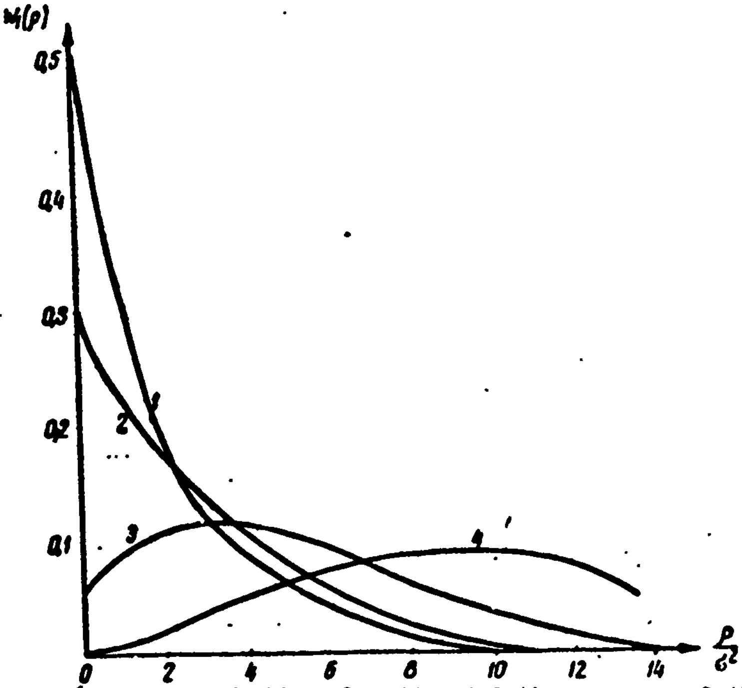


Fig. 60. Distribution function of the square of the envelope of a normal random process. 1. $\frac{\alpha}{\sigma} = 0$, 2. $\frac{\alpha}{\sigma} = 1$, 3. $\frac{\alpha}{\sigma} = 2$, 4. $\frac{\alpha}{\sigma} = 3$.

The distribution function corresponding to it is equal to

$$W_1(\rho)=\frac{1}{2\sigma^2}e^{-\frac{\rho}{2\sigma^2}},\quad \rho>0. \tag{8.20'}$$

This is the so-called exponential distribution.

The two-dimensional distribution function of the square of the envelope may also be obtained by the method of characteristic functions (cf. Sect. 7, Ch. VI).

Thus, in accordance with (6.81), the two-dimensional characteristic function of the square of the envelope of a stationary normal random process is equal to

$$\Theta_2(v_1, v_2, \tau)=\frac{2}{(2\pi\sigma^2)^2(1-R_0^2)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{iv_1(x_1^2+y_1^2)+iv_2(x_2^2+y_2^2)}\times e^{-\frac{x_1^2+y_1^2-2R_0(x_1x_2+y_1y_2)+x_2^2+y_2^2}{2\sigma^2(1-R_0^2)}}dx_1dx_2dy_1dy_2=$$

$$=\left[\frac{1}{2\pi\sigma^2\sqrt{1-R_0^2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i(v_1x_1^2+v_2x_2^2)}e^{-\frac{x_1^2+x_2^2-2R_0x_1x_2}{2\sigma^2(1-R_0^2)}}dx_1dx_2\right]^2, \tag{8.22}$$

i.e., to the square of the integral

$$\frac{1}{2\pi\sigma^2\sqrt{1-R_0^2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i(v_1x_1^2+v_2x_2^2)}e^{-\frac{x_1^2+x_2^2-2R_0x_1x_2}{2\sigma^2(1-R_0^2)}}dx_1dx_2=$$

$$=\frac{1}{\sqrt{1-2i\sigma^2(v_1+v_2)-4\sigma^4(1-R_0^2)v_1v_2}}. \tag{8.23}$$

A detailed computation of integral (8.23) is cited in Appendix V.

Consequently,

$$\Theta_2(v_1, v_2, \tau)=\frac{1}{1-2i\sigma^2(v_1+v_2)-4\sigma^4(1-R_0^2)v_1v_2}. \tag{8.24}$$

Effecting in (8.24) an inverse Fourier transformation (cf. Appendix VIII), we obtain a second distribution function of the envelope of a stationary random process, differing in no way from (8.19). The case where a regular signal is present may be marked out in an analogous manner. It is clear that for the investigation of the distribution functions of the square of the envelope, direct replacement of the variables in (8.9) is more simple than the employment of the method of characteristic functions. However, with more complex nonlinear transformations of the envelope $f_0(E)$, when a function inverse to f_0 cannot even exist, the method of characteristic

functions becomes one of the most effective procedures in solving the problem.

Having the expressions of the two-dimensional probability density of the square of the envelope, it is possible to find its correlation function, employing for this purpose the method of expansion in terms of orthogonal polynomials. As an example, let us determine the correlation function of the square of the envelope of a stationary normal random process, the two-dimensional distribution function of which is given by formula (8.19), and whose one-dimensional distribution function is equal to $\frac{1}{2\sigma^2} e^{-\frac{\rho}{2\sigma^2}}$ when $\rho > 0$. If this one-dimensional distribution function is taken as a weighting function over the interval of $(0,\infty)$, then to it will correspond the aggregate of orthogonal Laguerre polynomials $L_n^{(0)}\left(\frac{\rho}{2\sigma^2}\right)$ (cf. footnote on p. 306).

An expansion of the two-dimensional distribution function (8.19) into a series of these polynomials has the form of

$$\frac{1}{4\sigma^4(1-R_0^2)} e^{-\frac{\rho_1+\rho_2}{2\sigma^2(1-R_0^2)}} I_0\left[\frac{R_0\sqrt{\rho_1\rho_2}}{\sigma^2(1-R_0^2)}\right] = \frac{1}{4\sigma^4} e^{-\frac{\rho_1+\rho_2}{2\sigma^2}} \sum_{n=0}^{\infty} \frac{R_0^{2n}}{(n!)^2} L_n^{(0)}\left(\frac{\rho_1}{2\sigma^2}\right) L_n^{(0)}\left(\frac{\rho_2}{2\sigma^2}\right). \tag{8.25}$$

Employing (8.25), we represent the correlation function of the square of the envelope by the series

$$B_0(\tau) = \sum_{n=0}^{\infty} \frac{R_0^{2n}}{(n!)^2} \cdot \frac{1}{4\sigma^4} \int_0^{\infty}\int_0^{\infty} \rho_1\rho_2 L_n^{(0)}\left(\frac{\rho_1}{2\sigma^2}\right) L_n^{(0)}\left(\frac{\rho_2}{2\sigma^2}\right) \times e^{-\frac{\rho_1+\rho_2}{2\sigma^2}} d\rho_1 d\rho_2 = 4\sigma^4 \sum_{n=0}^{\infty} R_0^{2n} c_n^2, \tag{8.26}$$

where

$$c_n = \frac{1}{n!} \int_0^{\infty} x L_n^{(0)}(x) e^{-x} dx. \tag{8.27}$$

Employing the definition of the Laguerre polynomials (cf. footnote on p. 306), it is not difficult to compute integral (8.27):

$$c_n = \frac{(-1)^n}{n!}\int_0^\infty x e^{-x} e^x \frac{d^n}{dx^n}(x^n e^{-x})\,dx =$$

$$= \frac{(-1)^{n+1}}{n!}\int_0^\infty \frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x})\,dx =$$

$$= \frac{(-1)^{n+1}}{n!}\left[\frac{d^{n-2}}{dx^{n-2}}(x^n e^{-x})\right]_0^\infty \equiv 0, \text{ if } n \geqslant 2.$$

Consequently, in the series (8.26) only the first two coefficients, c_0 and c_2 differ from zero; these are computed independently, remembering that $L_0^{(0)}(x) = 1$ and $L_1^{(0)}(x) = 1 - x$

$$c_0 = \int_0^\infty x e^{-x}\,dx = 1,\ c_1 = \int_0^\infty x(1-x)e^{-x}\,dx = -1.$$

Thus, the correlation function of the square of the envelope of a stationary normal process is equal to

$$B_0(\tau) = 4\sigma^4[1 + R_0^2(\tau)]. \tag{8.28}$$

From a comparison of (8.28) and (7.63), it can be seen (as should have been expected) that the power spectrum of the square of the envelope coincides with the low-frequency part of the spectrum of a stationary normal random process after its square-law detection.

In the case under consideration an expression of the correlation function may be obtained directly from (8.24), if use is made of formula (5.16'). Differentiating $\Theta_2(v_1, v_2)$ first with respect to v_1, and then to v_2, we obtain from (8.24)

$$\frac{\partial^2\Theta_2}{\partial v_1 \partial v_2} = \frac{4\sigma^4(1-R_0^2)[1-2i\sigma^2(v_1+v_2)-4\sigma^4(1-R_0^2)v_1v_2]}{[1-2i\sigma^2(v_1+v_2)-4\sigma^4(1-R_0^2)v_1v_2]^3} +$$

$$+ \frac{[2i\sigma^2 + 4\sigma^4(1-R_0^2)v_2][4i\sigma^2 + 8\sigma^4(1-R_0^2)v_1]}{[1-2i\sigma^2(v_1+v_2)-4\sigma^4(1-R_0^2)v_1v_2]^3},$$

wherefrom

$$B_0(\tau) = -\left(\frac{\partial^2\Theta}{\partial v_1 \partial v_2}\right)_{v_1=0,\, v_2=0} =$$

$$= -4\sigma^4(1-R_0^2) + 8\sigma^4 = 4\sigma^4(1+R_0^2),$$

which does not differ from (8.28).

5. Statistical Criteria for Detection of Signals in Noise.

A distinctive feature of the contemporary theory of electrical signal transmission in communications, radar and telemechanics systems, is the fact that in this theory an evaluation of the influence of interference is not restricted to such a general criterion as the signal/noise ratio (cf. Sect. 8, Ch. VII), but employs the finer statistical properties of the processes, which make it possible to judge the authenticity of the data received. Whereas for calculation of the signal/noise ratio it is sufficient to have the power spectrum of the process at the output of the device from which the data is being picked up, an evaluation of the authenticity of received data by any statistical criterion will always require a knowledge of the multidimensional distribution functions of the process.

From these statistical positions let us examine the problem of the detection of a signal in noise.

Let the operator observe, on an indicator, a random-process envelope which may represent either noise alone, or the sum of a signal and noise. He does not know in advance whether a signal is present, and must decide this question on the basis of observations. Let the observations be fixed at a definite place, i.e., the problem being solved concerns the presence of a signal at a given point. The operator has made the following decision: he considers that the signal is present, if the voltage at the given point exceeds a certain arbitrary level u_o, and that it is absent in the contrary case. What is the probability that such a decision will yield the correct answer?

It is clear that an erroneous answer may be given in two mutually exclusive instances: 1) when the signal is absent, but the voltage exceeds the level u_o (event A), 2) when the signal is present, but the sum of the signal and the noise does not exceed the level u_o (event B).

The probability of event A, i.e., that two events will be combined, the absence of a signal and the exceeding by noise of the level u_o, is according to the rule of multiplication (cf. Sect. 3, Ch. I), equal to the a priori probability of the absence

of a signal, multiplied by the a posteriori probability of exceeding the level u_0 under the condition that a signal is absent.

The a priori probability q we shall take to be given, and the a posteriori probability of the noise exceeding level u_0 is not difficult to obtain from the first distribution function $W_1(r)$ of the noise envelope [cf., e.g., (8.3')]

$$P\{E>u_0\}=1-P\{E<u_0\}=1-\int_0^{u_0} w_1(r)\,dr=$$

$$=\int_{u_0}^{\infty} w_1(r)\,dr.$$

Then

$$P(A)=q\int_{u_0}^{\infty} w_1(r)\,dr. \tag{8.29}$$

The probability of event Б, i.e., that two events will be combined, as the presence of a signal and the failure to exceed level u_0, is according to the rule of multiplication equal to the a priori probability of the presence of a signal, multiplied by the a posteriori probability of the failure to exceed level u_0 under the condition that a signal is present.

The a priori probability of the presence of a signal is equal to

$$p=1-q,$$

and the a posteriori probability of the failure to exceed level u_0 is not difficult to obtain from the first distribution function of the signal-and-noise envelope [cf. e.g., (8.3)]

$$P\{E<u_0\}=\int_0^{u_0} W_1(R,\ u)\,dR.$$

Then

$$P(Б)=p\int_0^{u_0} W_1(R,\ u)\,dR. \tag{8.30}$$

Since events A and Б are mutually exclusive, on the basis of the rule of addition (cf. Sect. 2, Ch. I) we find the probability of an erroneous answer

$$P(A \text{ or } Б)=P(A)+P(Б)=q\int_{u_0}^{\infty} w_1(r)\,dr+$$

$$+p\int_0^{u_0} W_1(R,\ u)\,dR.$$

or

$$P(A \text{ or } Б) = 1 - \left(p \int_{u_0}^{\infty} W_1(R, u)\, dR + \right.$$

$$\left. + q \int_{0}^{u_0} w_1(r)\, dr \right).$$

Consequently the desired probability of the correct answer is equal to

$$P(u_0, u) = 1 - P(A \text{ or } Б) = p \int_{u_0}^{\infty} W_1(R, u)\, dR +$$

$$+ q \int_{0}^{u_0} w_1(r)\, dr. \tag{8.31}$$

The question, however, arises: what is the best way of selecting the arbitrary level u_0? It is clear, that if this level is chosen sufficiently high, the probability P(A) of the false detection of a signal will be small, but the probability P(Б) of missing a true signal may be considerable. Conversely, with a sufficiently low level of u_0 the probability of missing a signal will be small, but the probability of false detection may become considerable.

It is possible to formulate the problem of selecting the optimum magnitude of u_0, for which the probability $P(u_0, u)$ with given distribution functions of signal and noise is at a maximum. Computing from (8.31) the derivative $\frac{dP(u_0)}{du_0}$ and equating it to zero, we obtain the equation for determining the optimum level

$$q w_1(u_0) = p \dot{W}_1(u_0, u). \tag{8.32}$$

When p = q the optimum level is determined by the point of intersection of the distribution curve of noise with the curve of the joint distribution of signal and noise (Figure 61). As can be seen from the figure, with a strong signal the level u_0 must be chosen high, and with a weak signal this level approaches the mean-square noise voltage.

It is not difficult to notice an analogy with the problem cited in Sect. 4, Ch. I if the communication "yes" is made to correspond with the signal, and the lighting of the green lamp with the exceeding of the voltage on the indicator of level u_0.

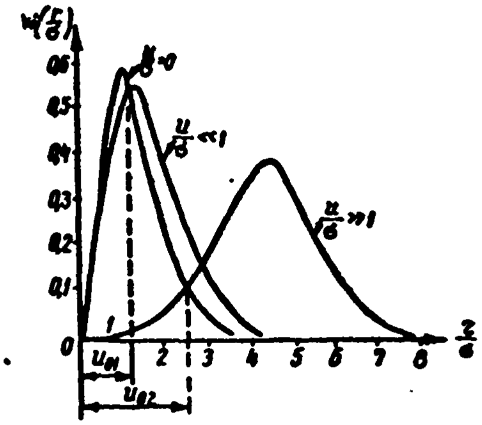


Fig. 61. Selection of optimum level u_0.

In the example under consideration the observer is restricted to two possible answers: "signal present" or "no signal", with the probability of the correct answer being determined. With the aid of Bayes' formula (1.16) it is possible to determine the a posteriori probability of the presence of a signal, under the condition that the observed voltage exceed the level u_0. A similar approach to the problem of signal detection in noise is developed in work [7].

Formula (8.31) is extended to the case where the operator must decide the question of the presence or absence of a signal on the basis of N observations of the envelope of a random process. The aggregate of N observations of the random voltage $\xi_1, \xi_2, \ldots \xi_N$ may be represented by a point in N-dimensional space. The operator arbitrarily divides this space into two ranges G_c and $G_{ш}$, and considers that the signal is present if point $K(\xi_1, \xi_2, \ldots \xi_N)$ falls into range G_c and that it is absent, if the point K falls into range $G_{ш}$.

Let $w_N(r_1, \ldots r_N, \tau_1, \ldots \tau_{N-1})$ and $W_N(R_1, \ldots R_N, \tau_1 \ldots \tau_{N-1}, u)$ be N-dimensional distribution functions respectively of the noise envelope and of the signal-and-noise envelope. Then the probability of a correct answer is determined according to the formula

$$P_N(G_c, u) = \\ = p\int\limits_{G_c}\ldots\int W_N(R_1, \ldots R_N, \tau_1, \ldots \tau_{N-1}, u)\,dR_1\ldots dR_N + \\ + q\int\limits_{G_{ш}}\ldots\int w_N(r_1, \ldots r_N, \tau_1, \ldots \tau_{N-1})\,dr_1\ldots dr_N. \qquad (8.33)$$

In this case it is also possible to formulate the problem of selecting the

optimum surface S_0 which divides the space into the ranges G_c and G_{ω} so, that the probability $P_N(G_c, u)$ is at a maximum with given distribution functions of signal and noise. The equation for this surface is determined from the relationship

$$qw_N(z_1, z_2, \ldots z_N) = pW_N(z_1, z_2 \ldots z_N, u),$$

which for N = 1 was cited above. For two observations when $p = q = \frac{1}{2}$ it permits also a geometrical interpretation. The optimum line dividing the plane (ξ_1, ξ_2) into two ranges is a projection of the line of intersection of the noise-distribution surface with the joint signal-and-noise distribution surface.

If the time intervals τ_i between the observations are sufficiently great, then the magnitudes $\xi_1, \xi_2, \ldots \xi_N$ may be considered mutually independent.

Then

$$W_N(R_1, \ldots R_N, \tau_1, \ldots \tau_{N-1}, u) = W_1(R_1, u) \ldots W_1(R_N, u)$$
$$w_N(r_1, \ldots r_N, \tau_1, \ldots \tau_{N-1}) = w_1(r_1) \ldots w_1(r_N).$$

In this case, for a weak signal the equation for the optimum surface S_0 has the form of

$$\xi_1^2 + \xi_2^2 + \ldots + \xi_N^2 = 2N\sigma^2,$$

i.e., constitutes a hypersphere. Representing this equation in the form of

$$\eta_N = \frac{1}{N} \sum_{i=1}^{N} \xi_i^2 = 2\sigma^2,$$

we conclude that in the event of a weak signal the operator must compare the mean-square value of N observations with the mean-square value of the noise. If $\eta_N > 2\sigma^2$, then on the basis of N observations he draws his conclusion as to the presence of a signal.

If N is sufficiently great, then it is possible to show that the probability of a correct answer with $p = q = \frac{1}{2}$ is in the optimum case computed according to the formula

$$P_N(s_0, u) = F\left(\frac{\sqrt{N}}{2} \cdot \frac{u^2}{2\sigma^2}\right). \qquad (8.34)$$

where F(x) is the Laplace function.

Formula (8.34) makes it possible to indicate the minimum number of measurements necessary to obtain a correct answer with the required reliability for a given $\frac{u^2}{2\sigma^2}$. The dependence curve of function P_N on $\frac{u^2}{2\sigma^2}$ is shown in Figure 62.

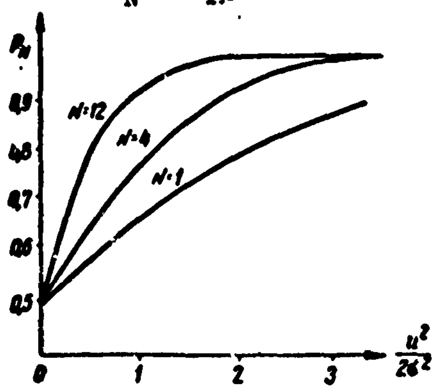


Fig. 62. Probability curve for right answer in optimum case.

Above there has been examined a statistical criterion for the detection of a signal in noise, which provides the maximum probability of a right answer with one or with several observations. This criterion is called the criterion of "the ideal observer".

Other statistical criteria are, however, also possible. For instance, in some cases it is important that the probability P(Б) of missing an existing signal does not exceed some constant K. In this case with a given number of N observations it is necessary to select a criterion for which the probability P(A) of false detection is at a minimum under the condition that $P(Б) \leqslant K$ (the Niemann-Pearson criterion).

There can be selected a statistical criterion of the "successive observer", which with given probabilities of missing and of false detection will make it possible to reduce the number of observations N. According to this criterion an N-dimensional range of possible signal and noise values is broken up into three: $G_ш$, G_c, and an intermediate one. If, when $n < N$, the aggregate of observed values falls into the intermediate range, then the following observation is made, and so on until this aggregate falls either into range $G_ш$ or range G_c, after which a decision is made as to the presence or absence of a signal.

A comparison of the indicated three statistical criteria for the detection of a signal in noise is represented schematically in Figure 63 (in the general case ranges

$G_ш$ and G_c are not one-dimensional). A detailed presentation of the employment of statistical methods in the problem of signal detection in noise will be found by the reader in [3], [6].

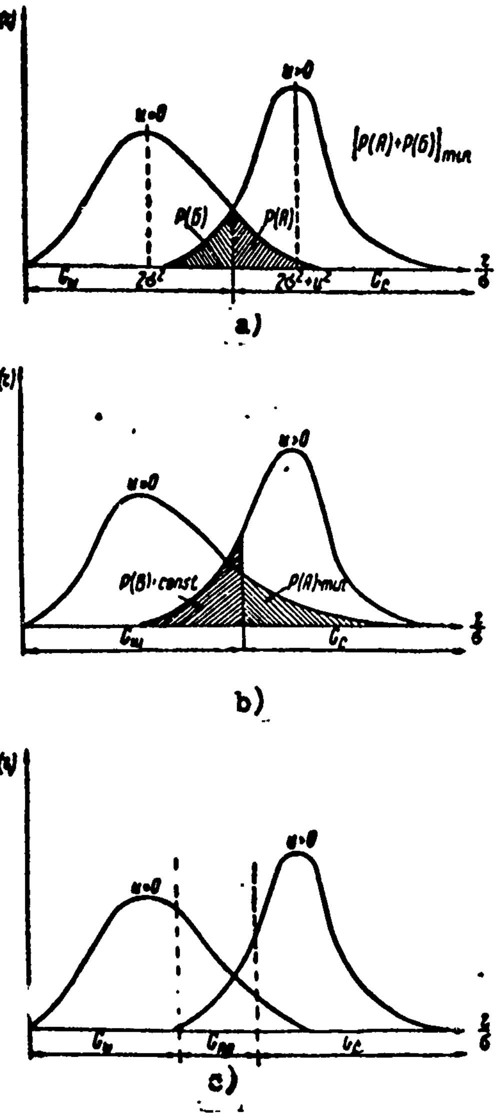


Fig. 63. Comparison of statistical criteria for detection of signal in noise. a) ideal observer; b) Niemann-Pearson observer; c) successive observer.

6. One-dimensional Distribution Function of Phase

We pass to the study of the statistical characteristics of the phase of a normal random process. In accordance with the general formula (6.80), the one-dimensional

distribution function of the phase of random process (8.1') is equal to

$$W_1(\Theta, t) = \frac{1}{2\pi\sigma^2}\int_0^\infty r e^{-\frac{(r\cos\Theta - u)^2 + (r\sin\Theta - v)^2}{2\sigma^2}} dr.$$

Effecting elementary transformations, we obtain

$$W_1(\Theta, t) = \frac{1}{2\pi\sigma^2} e^{-\frac{u^2+v^2}{2\sigma^2}} \int_0^\infty r e^{-\frac{r^2 - 2r(u\cos\Theta + v\sin\Theta)}{2\sigma^2}} dr. \tag{8.35}$$

Designating

$$u^2(t) + v^2(t) = \alpha^2(t), \quad \Theta_s(t) = \operatorname{arctg}\frac{v(t)}{u(t)},$$

we rewrite (8.35) in the form of

$$W_1(\Theta, t) = \frac{1}{2\pi\sigma^2} e^{-\frac{\alpha^2}{2\sigma^2}} \int_0^\infty r e^{-\frac{r^2 - 2r\alpha\cos(\Theta - \Theta_s)}{2\sigma^2}} dr. \tag{8.36}$$

The integral in (8.36) is calculated by means of a replacement of the variable of integration

$$W_1(\Theta, t) = \frac{1}{2\pi\sigma^2} e^{-\frac{\alpha^2}{2\sigma^2}} e^{\frac{\alpha^2}{2\sigma^2}\cos^2(\Theta - \Theta_s)} \int_0^\infty r e^{-\frac{[r - \alpha\cos(\Theta - \Theta_s)]^2}{2\sigma^2}} dr =$$

$$= \frac{1}{2\pi\sigma^2} e^{-\frac{\alpha^2}{2\sigma^2}\sin^2(\Theta - \Theta_s)} \int_{-\alpha\cos(\Theta - \Theta_s)}^\infty [r + \alpha\cos(\Theta - \Theta_s)] e^{-\frac{r^2}{2\sigma^2}} dr,$$

wherefrom, introducing the Laplace function (cf. Sect. 7, Ch. I), we find

$$W_1(\Theta, t) = \frac{1}{2\pi} e^{-\frac{\alpha^2}{2\sigma^2}} + \frac{\alpha\cos(\Theta - \Theta_s)}{\sigma\sqrt{2\pi}} F\left[\frac{\alpha}{\sigma}\cos(\Theta - \Theta_s)\right] e^{-\frac{\alpha^2}{2\sigma^2}\sin^2(\Theta - \Theta_s)} \tag{8.37}$$

$$\Theta_s - \pi \leqslant \Theta \leqslant \Theta_s + \pi.$$

If the signal constitutes a harmonic vibration with a frequency of ω_0 and an amplitude of u_0, then $\alpha = u_0$, $\Theta_s = 0$, and from (8.37) there follows

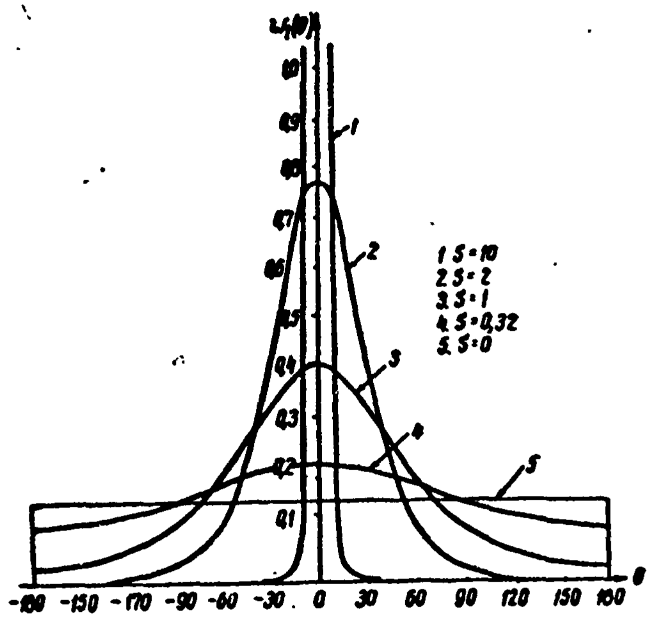
$$W_1(\Theta) = \frac{1}{2\pi} e^{-\frac{s^2}{2}} + \frac{s\cos\Theta}{\sqrt{2\pi}} F(s\cos\Theta) e^{-\frac{s^2\sin^2\Theta}{2}}, \tag{8.38}$$

$$-\pi \leqslant \Theta \leqslant \pi$$

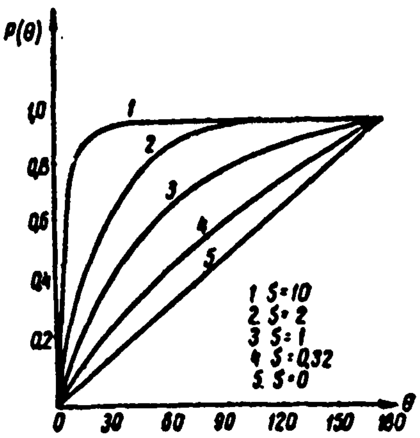
whereby $s = \frac{u_0}{\sigma}$ is represented the ratio of the amplitude of the signal to the mean-square value of the noise. It is obvious, that in a fixed moment of time distribution (8.37) has the same form as (8.38), if only the origin of the coordinates is transferred to the point $\Theta = \Theta_s$ and there is designated $s = \frac{\alpha(t)}{\sigma}$.

Let us examine distribution (8.38) more closely. Figure 64,a shows a set of

phase distribution curves, computed according to formula (8.38).



a)



b)

Fig. 64. Distribution function of phase of normal random process:

a) probability density $W_1(\Theta)$,

b) integral function $P(\Theta)$ (cf. p. 326 and further).

As can be seen from (8.38) and from Figure 64,a the distribution function $W_1(\Theta)$ is even along Θ , i.e., the distribution curve corresponding to it is symmetrical with respect to the ordinate axis. When s = 0

$$W_1(\theta)=\frac{1}{2\pi},\ -\pi<\Theta<\pi.$$

which corresponds to a uniform phase distribution for pure noise. With the presence of a signal, as s increases, the probability density for $\Theta = 0$ also increases, being equal to

$$W_1(0)=\frac{1}{2\pi}e^{-\frac{s^2}{2}}+\frac{sF(s)}{\sqrt{2\pi}}\geqslant\frac{1}{2\pi}. \tag{8.39}$$

At the boundaries of the interval $(-\pi, \pi)$ the probability density is less than $\frac{1}{2\pi}$ and, as s increases, tends toward zero

$$W_1(\pm\pi)=\frac{1}{2\pi}e^{-\frac{s^2}{2}}-\frac{s}{\sqrt{2\pi}}[1-F(s)]\leqslant\frac{1}{2\pi}. \tag{8.40}$$

When $\Theta = \pm\frac{\pi}{2}$ the probability density is equal to

$$W_1\left(\pm\frac{\pi}{2}\right)=\frac{1}{2\pi}e^{-\frac{s^2}{2}}.$$

It is easy to see, that

$$W_1(\pi-\Theta)=W_1(\Theta)-\frac{s\cos\Theta}{\sqrt{2\pi}}e^{-\frac{s^2\sin^2\Theta}{2}},\ \left(\Theta<\frac{\pi}{2}\right), \tag{8.41}$$

i.e., that the probability densities of two points, symmetrically situated with respect to the $\pm\frac{\pi}{2}$ axis at a distance of $\frac{\pi}{2}-\Theta$, differ by $\frac{s\cos\Theta}{\sqrt{2\pi}}e^{-\frac{s^2\sin^2\Theta}{2}}$.

When $s \ll 1$, i.e., with a signal amplitude much smaller than the mean-square value of the noise (weak signal), from (8.38), expanding the right part into an exponential series in terms of s and restricting ourselves to terms of the second order of smallness, we obtain

$$W_1(\Theta)=\frac{1}{2\pi}+\frac{s\cos\Theta}{2\sqrt{2\pi}}+\frac{s^2\cos 2\Theta}{4\pi},\ s\ll 1. \tag{8.42}$$

In this case, with an accuracy to quantities no greater than s^2, the point of intersection of $W_1(\Theta)$ with the line $W_1 = \frac{1}{2\pi}$ is equal to

$$\Theta^*=\pm\arc\cos\frac{2s}{\sqrt{2\pi}}\approx\pm\left(\frac{\pi}{2}-\frac{2s}{\sqrt{2\pi}}\right). \tag{8.43}$$

Therefore when $|\Theta|<\Theta^*$ $W_1(\Theta)>\frac{1}{2\pi}$, and when $|\Theta|>\Theta^*$ $W_1(\Theta)<\frac{1}{2\pi}$. from (8.42) it follows, that with a very weak signal the phase distribution function constitutes a cosinusoid, displaced along the ordinate axis by the amount of $\frac{1}{2\pi}$, with an amplitude of $\frac{s}{2\sqrt{2\pi}}$.

If $s\cos\Theta > 2{,}5\left(\Theta < \frac{\pi}{2},\ s > 2{,}5\right)$, i.e., with a signal amplitude much greater than the mean-square value of the noise (strong signal), from (8.38) we obtain

$$W_1(\Theta) \sim \frac{s\cos\Theta}{\sqrt{2\pi}} e^{-\frac{s^2\sin^2\Theta}{2}}. \tag{8.44}$$

When $\Theta \geqslant \frac{\pi}{2}$ $(s > 2.5)$ it may be considered that $W_1(\Theta) \equiv 0$, and for small values of Θ

$$W_1(\Theta) \approx \frac{s}{\sqrt{2\pi}} e^{-\frac{s^2\Theta^2}{2}}, \tag{8.45}$$

i.e., the distribution law for a phase is normal with a dispersion of $\frac{1}{s^2} = \frac{\sigma^2}{U_0^2}$, equal to the noise-signal ratio in the initial normal random process.

As s increases, the phase probability density approaches the delta-function (cf. Appendix IV)

$$\lim_{s\to\infty} W_1(\Theta) = \delta(\Theta), \tag{8.46}$$

which characterizes the distribution of the signal phase (assumed to be zero) in the absence of noise.

We compute the numerical characteristics (distribution moments) for function (8.38):

$$m_n\{\varphi\} = \int_{-\pi}^{\pi} \Theta^n W_1(\Theta)\, d\Theta. \tag{8.47}$$

For computing the integral it is useful first to represent $W_1(\Theta)$ by a Fourier series along Θ over the interval $(-\pi, \pi)$. For this it is sufficient to expand the integrand function in (8.35) into a Fourier series in terms of Θ, and to make use of the well-known equality from the theory of Bessel functions (cf., e.g. G. N. Vatson, "Teoriya besselevykh funktsiy" (G. N. Watson, A Treatise on the Theory of Bessel Functions) For. Lit. Publ. Hse., 1949, p. 31)

$$e^{sz\cos\Theta} = \sum_{n=-\infty}^{\infty} I_n(sz) e^{in\Theta} = \tag{8.48}$$

$$= I_0(sz) + 2\sum_{n=1}^{\infty} I_n(sz)\cos n\Theta.$$

Then we obtain the sum of the integrals

$$W_1(\Theta)=\frac{1}{2\pi}e^{-\frac{s^2}{2}}\int_0^\infty I_0(sz)ze^{-\frac{z^2}{2}}dz+$$

$$+\frac{1}{\pi}e^{-\frac{s^2}{2}}\sum_{n=1}^\infty\int_0^\infty I_n(sz)ze^{-\frac{z^2}{2}}dz\cos n\Theta.$$

Employing an expression of the integrals obtained, in terms of degenerate hypergeometric functions (cf. Appendix VI, we find the desired resolution of $W_1(\Theta)$ into a Fourier series

$$W_1(\Theta)=\frac{1}{2\pi}+\sum_{k=1}^\infty a_k\cos k\Theta, \tag{8.49}$$

where there is designated

$$a_k=\frac{\Gamma\left(1+\frac{k}{2}\right)s^k}{\pi 2^{\frac{k}{2}}k!}{}_1F_1\left(\frac{k}{2},\ k+1,\ -\frac{s^2}{2}\right). \tag{8.50}$$

Substituting (8.49) into (3.47), and taking into account that the product Θ^n $\cos k\Theta$ is an even function when n is even and an odd function when n is odd, we obtain

$$m_{2r}=\frac{\pi^{2r}}{2r+1}+\sum_{k=1}^\infty a_k\int_{-\pi}^{\pi}\Theta^{2r}\cos k\Theta\,d\Theta,$$

$$m_{2r+1}=0,\ r=0,\ 1,\ 2,\dots \tag{8.51}$$

Since the mean value of $m_1 = 0$, the random-phase dispersion σ_φ^2 coincides with the second distribution moment, and from (8.51) we have

$$\sigma_\varphi^2=m_2=\frac{\pi^2}{3}+\sum_{k=1}^\infty a_k\int_{-\pi}^{\pi}\Theta^2\cos k\Theta d\Theta,$$

or, after computing the integral,

$$\sigma_\varphi^2=\frac{\pi^2}{3}+4\pi\sum_{k=1}^\infty(-1)^k\frac{a_k}{k^2}. \tag{8.52}$$

With a weak signal it is possible to restrict one's self to the first term of a series (8.52), and then

$$\sigma_\varphi^2=\frac{\pi^2}{3}-4\pi a_1=\frac{\pi^2}{3}-s\sqrt{2\pi}{}_1F_1\left(\frac{1}{2},\ 2,\ -\frac{s^2}{2}\right),$$

or

$$\sigma_\varphi^2\approx\frac{\pi^2}{3}-s\sqrt{2\pi},\quad s\ll 1. \tag{8.53}$$

For a strong signal, the phase dispersion diminishes with the increase in signal amplitude, with

$$\sigma_\varphi^2 \sim \frac{1}{s^2}. \tag{8.54}$$

Equality (8.54) is obtained from (8.52), if use is made of an asymptotic series for the degenerate hypergeometric functions (cf. Appendix VI).

7. Errors in Measuring Phase of Harmonic Vibration due to Presence of Noise.

At the present time, increasingly widespread application in experimental engineering is being made of the so-called phase method of measurements, linked with the transformation of the process under examination into a process of frequency change or of alternating-voltage phase.

It is well known that, in amplitude measurements, instrument sensitivity is limited by fluctuation noise. In phase measurements, fluctuation noise leads to variations in the magnitude of the phase difference between two signals, which is being measured. In this connection interest is afforded by the question of errors in phase measurement, which are caused by the presence of noise.

Let there be measured the phase of a high-frequency harmonic vibration with an amplitude of u_0 in the presence of narrow-band, normal, stationary noise. What is the probability that the error in the measurement of the fluctuation phase does not exceed some fixed magnitude Θ? It is obvious, that this probability is simply expressed in terms of the integral distribution function

$$P\{|\varphi|<\Theta\}=\int_{-\Theta}^{\Theta} W_1(\Theta)\,d\Theta=2\int_{0}^{\Theta} W_1(\Theta)\,d\Theta, \tag{8.55}$$

where $W_1(\Theta)$ is determined by formula (8.38).

Let us introduce the abbreviated designation

$$P\{|\varphi|<\Theta\}=P(\Theta). \tag{8.56}$$

Substituting (8.38) into (8.55) we obtain

$$P(\Theta)=\frac{\Theta}{\pi}e^{-\frac{s^2}{2}}+\frac{2s}{\sqrt{2\pi}}\int_{0}^{\Theta}\cos\Theta\, F(s\cos\Theta)\, e^{-\frac{s^2\sin^2\Theta}{2}}\,d\Theta. \tag{8.57}$$

The integral in the right part of (8.57) is investigated in Appendix IX. Employing formulas (9) and (12) obtained in this appendix, we find

$$P(\Theta)=F(s\sin\Theta)-\frac{1}{\pi}\left(\frac{\pi}{2}-\Theta\right)+2V(s\sin\Theta,\ s\cos\Theta),\qquad 0<\Theta<\frac{\pi}{2}, \tag{8.58}$$

$$P(\Theta)=F(s\sin\Theta)+\frac{1}{\pi}\left(\Theta-\frac{\pi}{2}\right)-2V(s\sin\Theta,-s\cos\Theta),\qquad \frac{\pi}{2}<\Theta<\pi. \tag{8.58'}$$

The function V(h, q) has been tabulated by Nicholson (cf. Biometrica, V. 38, 1943). These tables have been employed to plot the set of curves for the integral law of the distribution of phase $P(\Theta)$, which is shown in Figure 64,b.

We note, that from (8.58) there directly follows

$$P(\pi-\Theta)+P(\Theta)=2F(s\sin\Theta). \tag{8.59}$$

From (8.59) we find when $\Theta = \frac{\pi}{2}$

$$P\left(\frac{\pi}{2}\right)=F(s). \tag{8.59'}$$

If $s\cos\Theta \gg 3$, then function V permits the following asymptotic representation:

$$2V(s\sin\Theta,\ s\cos\Theta)\sim F(s\sin\Theta)-\frac{1}{2}-\frac{\Theta}{\pi}. \tag{8.60}$$

Substituting (8.60) into (8.58), we obtain an asymptotic formula for the probability of $P(\Theta)$, which is valid when $s\cos\Theta \gg 3$.

$$P(\Theta)=2F(s\sin\Theta)-1. \tag{8.61}$$

If $s \ll 1$ (weak signal), then, taking (8.42) into account, we find

$$P(\Theta)\approx\frac{\Theta}{\pi}+\frac{s}{\sqrt{2\pi}}\sin\Theta+\frac{s^2}{4\pi}\sin 2\Theta. \tag{8.62}$$

8. Two-dimensional Phase-distribution Function

In accordance with (6.80), the two-dimensional distribution function of the phase of the normal random process (8.1) is equal to

$$W_2(\Theta_1,\Theta_2,\tau,t)=\frac{1}{(2\pi\sigma^2)^2(1-R_0^2)}\int_0^\infty\!\!\int_0^\infty r_1r_2\,dr_1dr_2\times$$
$$\times\exp\left\{-\frac{1}{2\sigma^2(1-R_0^2)}\{(r_1\cos\Theta_1-u_1)^2+(r_1\sin\Theta_1-v_1)^2+\right. \tag{8.63}$$

$$+(r_2\cos\Theta_2-u_2)^2+(r_2\sin\Theta_2-v_2)^2-$$
$$-2R_0[(r_1\cos\Theta_1-u_1)(r_2\cos\Theta_2-u_2)+$$
$$+(r_1\sin\Theta_1-v_1)(r_2\sin\Theta_2-v_2)]\Big\}\Big\},$$
$$-\pi\leqslant\Theta_1\leqslant\pi,\quad -\pi\leqslant\Theta_2\leqslant\pi, \tag{8.63 (Cont'd)}$$

where
$$u_1=u(t),\quad u_2=u(t+\tau),\quad v_1=v(t),\quad v_2=v(t+\tau).$$

When u = v = 0, which corresponds to a two-dimensional distribution of the phases of a stationary normal random process, there follows from (8.63)

$$W_2(\Theta_1,\Theta_2,\tau)=\frac{1}{4\pi^2\sigma^4(1-R_0^2)}\int_0^\infty\int_0^\infty r_1r_2 e^{-\frac{r_1^2+r_2^2-2R_0r_1r_2\cos(\Theta_1-\Theta_2)}{2\sigma^2(1-R_0^2)}}dr_1dr_2. \tag{8.64}$$

A computation of integral (8.64), cited in [1], [4], leads to the expression

$$W_2(\Theta_1,\Theta_2,\tau)=\frac{1-R_0^2}{4\pi^2}\left[\frac{1}{1-y^2}+y\frac{\frac{\pi}{2}+\arcsin y}{(1-y^2)^{3/2}}\right], \tag{8.65}$$
$$-\pi\leqslant\Theta_1\leqslant\pi,\quad -\pi\leqslant\Theta_2\leqslant\pi,$$

where

$$y=R_0\cos(\Theta_1-\Theta_2). \tag{8.66}$$

When $\tau\to\infty$, $R_0(\tau)\to 0$, $y\to 0$, and from (8.65) there follows

$$W_2(\Theta_1,\Theta_2)=W_1(\Theta_1)W_1(\Theta_2)=\frac{1}{4\pi^2}:$$

It is evident from (8.66) that when $\Theta_1-\Theta_2=$ const, the distribution function $W_2(\Theta_1,\Theta_2,\tau)$ = const, i.e., the intersections of the probability surface $W_2(\Theta_1,\Theta_2,)$ by the plane $\Theta_1-\Theta_2=0$, and by those parallel to it, are straight lines, parallel to the surface (Θ_1,Θ_2) and, consequently, the probability surface of the phase of pure noise is a ruled surface*. Besides that, function $W_2(\Theta_1,\Theta_2,\tau)$ is even along $\Theta_1-\Theta_2$, i.e., the plane $\Theta_1=\Theta_2$ is the plane

* As is well known, a ruled surface is formed by the movement of a straight line (the generatrix) as it slides along some curve (the directrix). The equation of the directrix in the case at hand is provided by formula (8.65).

of symmetry of the distribution surface.

A change in the level of the location of the indicated straight lines of the constant probability density is determined by the intersections of surface $W_2(\Theta_1, \Theta_2, \tau)$ by the set of parallel planes $\Theta_1 + \Theta_2$ = const, which is orthogonal to the set of planes $\Theta_1 - \Theta_2$ = const. One of these intersections (by the plane $\Theta_1 + \Theta_2 = \Theta$) is shown in Figure 65. The maximum probability density, corresponding to $\Theta_1 = \Theta_2$, is equal to

$$W_{2max}(\tau) = \frac{1}{4\pi^2}\left[1 + \left(\frac{\pi}{2} + \arcsin R_0\right)\frac{R}{\sqrt{1-R_0^2}}\right]. \tag{8.67}$$

From (8.67), it follows that $W_{2\,max}(\tau) \geqslant \frac{1}{4\pi^2}$, i.e., a correlation between the phases increases the probability density of $\Theta_1 = \Theta_2$, this density increasing limitlessly when $\tau \to 0$ ($R_0 \to 1$). Furthermore, if between the correlated phases $\varphi(t)$ and $\varphi(t+\tau)$ there takes place a constant displacement, equal to $\frac{\pi}{2}$, then from (8.65) there follows $W_2^*(\tau) = \frac{1-R_0^2}{4\pi^2}$, i.e., in this case the probability density is less than in the absence of correlation by the quantity $\frac{R_0^2}{4\pi^2}$. When $\tau \to 0$ ($R_0 \to 1$) the probability density of a displacement of $\frac{\pi}{2}$ tends toward zero.

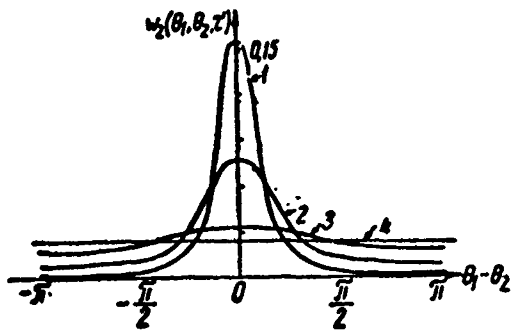


Fig. 65. Intersection of phase-distribution surface of noise by the plane $\Theta_1 = -\Theta_2$.

1) $R_0 = 0.9$; 2) $R_0 = 0.7$; 3) $R_0 = 0.2$; 4) $R_0 = 0$.

The minimum density, corresponding to $\Theta_1 = \pi + \Theta_2$, is equal to

$$W_{2min}(\tau) = \frac{1}{4\pi^2}\left(1 - \arccos R_0 \frac{R_0}{\sqrt{1-R_0^2}}\right). \tag{8.67'}$$

From (8.67') it follows that $W_{2\,min}(\tau) \leqslant \frac{1}{4\pi^2}$, i.e., correlation between the phases diminishes the probability density of $\Theta_1 = \Theta_2 + \pi$, this density tending toward zero when $\tau \to 0$ ($R_0 \to 1$).

From (8.65) it is not difficult to obtain the distribution function of the phase difference $\varphi(t+\tau)-\varphi(t)$

$$W_1(\theta)=\frac{1-R_0^2}{2\pi}\left[\frac{1}{1-y^2}+y\frac{\frac{\pi}{2}+\arcsin y}{(1-y^2)^{3/2}}\right]. \tag{8.68}$$

The two-dimensional distribution function of the phase of a normal process, for the case when there is present the signal u(t) $\cos\omega_0 t$, has the following form:

$$\begin{aligned}W_2(\Theta_1,\Theta_2,\tau,t)&=\frac{1-R_0^2}{4\pi^2}e^{-\frac{(u_1^2+u_2^2-2R_0u_1u_2)}{2\sigma^2(1-R_0^2)}}e^{-\frac{a_1^2+a_2^2}{2}}\times\\&\times\sum_{m=0}^{\infty}\left[2R_0\cos(\Theta_2-\Theta_1)\right]^m\cdot\frac{1}{m!}\left\{\frac{a_1}{\sqrt{2}}\Gamma\left(\frac{m+3}{2}\right)\times\right.\\&\times{}_1F_1\left(-\frac{m}{2},\frac{3}{2},-\frac{a_1^2}{2}\right)+\Gamma\left(\frac{m}{2}+1\right)\times\\&\left.\times{}_1F_1\left(-\frac{m+1}{2},\frac{1}{2},-\frac{a_1^2}{2}\right)\right\}\cdot\left\{\frac{a_2}{\sqrt{2}}\Gamma\left(\frac{m+3}{2}\right)\times\right.\\&\times{}_1F_1\left(-\frac{m}{2},\frac{3}{2},-\frac{a_2^2}{2}\right)+\Gamma\left(\frac{m}{2}+1\right)\times\\&\left.\times{}_1F_1\left(-\frac{m+1}{2},\frac{1}{2},-\frac{a_2^2}{2}\right)\right\},\\&-\pi\leqslant\Theta_1\leqslant\pi,\quad -\pi\leqslant\Theta_2\leqslant\pi,\end{aligned} \tag{8.69}$$

where

$$a_1=\frac{u_1-R_0u_2}{\sigma\sqrt{1-R_0^2}}\cos\Theta_1,\quad a_2=\frac{u_2-R_0u_1}{\sigma\sqrt{1-R_0^2}}\cos\Theta_2,$$
$$u_1=u(t),\quad u_2=u(t+\tau). \tag{8.70}$$

Formula (8.69) is sufficiently cumbersome*. In certain cases, it is more convenient for the analysis to employ a formula which results, if the two-dimensional phase-distribution function is represented by a multiple Fourier series in terms of variables Θ_1 and Θ_2. For this in integral (8.63) we assume v $\equiv$ 0, go over to the variables $z_1=\frac{r_1}{\sigma}$, $z_2=\frac{r_2}{\sigma}$ and employ formula (8.48). Then we obtain

$$\begin{aligned}W_2(\Theta_1,\Theta_2,\tau,t)&=\frac{1}{4\pi^2(1-R_0^2)}e^{-\frac{u_1^2+u_2^2-2R_0u_1u_2}{2\sigma^2(1-R_0^2)}}\times\\&\times\int_0^\infty\int_0^\infty z_1z_2e^{-\frac{z_1^2+z_2^2}{2(1-R_0^2)}}\sum_{r=-\infty}^{\infty}I_r\left(\frac{R_0z_1z_2}{1-R_0^2}\right)e^{ir(\Theta_2-\Theta_1)}\times\\&\times\sum_{n=-\infty}^{\infty}I_n\left[\frac{u_1-R_0u_2}{\sigma(1-R_0^2)}z_1\right]e^{in\Theta_1}\sum_{m=-\infty}^{\infty}I_m\left[\frac{u_2-R_0u_1}{\sigma(1-R_0^2)}z_2\right]e^{im\Theta_2}dz_1dz_2.\end{aligned}$$

* A certain simplification of formula (8.69) may be attained by employing relationship (8) of Appendix VI.

Changing the order of summation and integration and introducing designations for the expansion coefficients

$$A_{rnm}(\tau, t)=\frac{1}{4\pi^2(1-R_0^2)}e^{-\frac{u_1^2+u_2^2-2R_0u_1u_2}{2\sigma^2(1-R_0^2)}}\int\limits_0^\infty\int\limits_0^\infty z_1z_2 \times$$

$$\times I_r\left(\frac{R_0z_1z_2}{1-R_0^2}\right)I_n\left[\frac{u_1-R_0u_2}{\sigma(1-R_0^2)}z_1\right]I_m\left[\frac{u_2-R_0u_1}{\sigma(1-R_0^2)}z_2\right]e^{-\frac{z_1^2+z_2^2}{2(1-R_0^2)}}dz_1dz_2, \tag{8.71}$$

we obtain the desired expansion of the two-dimensional phase-distribution function into a multiple Fourier series

$$W_2(\Theta_1, \Theta_2, \tau, t)=\sum_{r=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}A_{rnm}\cdot e^{i[(r+n)\Theta_1+(m-r)\Theta_2]},$$
$$-\pi\leqslant\Theta_1\leqslant\pi, \quad -\pi\leqslant\Theta_2\leqslant\pi. \tag{8.72}$$

If the signal is absent ($u_1 = u_2 = 0$), then from (8.72) we obtain the expansion into a Fourier series of the two-dimensional distribution function of the phase of a stationary normal random process

$$W_2(\Theta_1, \Theta_2, \tau)=\sum_{r=-\infty}^{\infty}A_re^{ir(\Theta_1-\Theta_2)},$$
$$-\pi\leqslant\Theta_1\leqslant\pi, \quad -\pi\leqslant\Theta_2\leqslant\pi, \tag{8.73}$$

where

$$A_r(\tau)=A_{r00}(\tau)=\frac{1}{4\pi^2(1-R_0^2)}\int\limits_0^\infty\int\limits_0^\infty z_1z_2 \times$$
$$\times I_r\left(\frac{R_0z_1z_2}{1-R_0^2}\right)e^{-\frac{z_1^2-z_2^2}{2(1-R_0^2)}}dz_1dz_2. \tag{8.74}$$

Since $A_r = A_{-r}$, (8.73) may be rewritten in the form of

$$W_2(\Theta_1, \Theta_2, \tau)=A_0+2\sum_{r=1}^{\infty}A_r\cos r(\Theta_1-\Theta_2). \tag{8.75}$$

The coefficients A_r are computed by the expansion of function $I_r\left(\frac{R_0z_1z_2}{1-R_0^2}\right)$ into the exponential series

$$A_r=\frac{1}{4\pi^2}\sum_{n=0}^{\infty}\frac{R_0^{r+2n}}{(1-R_0^2)^{r+1+2n}}\times$$

$$\times\frac{1}{n!(n+r)!2^{r+2n}}\int\limits_0^\infty\int\limits_0^\infty z_1^{r+1+2n}z_2^{r+1+2n}e^{-\frac{z_1^2+z_2^2}{2(1-R_0^2)}}dz_1dz_2= \tag{8.76}$$

$$=\frac{1-R_0^2}{4\pi^2}\sum_{n=0}^{\infty}\frac{R_0^{r+2n}}{n!(n+r)!\,2^{r+2n}}\left(\int_0^{\infty}x^{r+1+2n}e^{-\frac{x^2}{2}}dx\right)^2=$$

$$=\frac{1-R_0^2}{4\pi^2}\sum_{n=0}^{\infty}\frac{\Gamma^2\left(n+1+\frac{r}{2}\right)}{n!(n+r)!}R_0^{r+2n}. \qquad (8.76) \text{ (Cont'd)}$$

From (8.76) it follows that the constant term in (8.75) is always equal to $\frac{1}{4\pi^2}$:

$$A_0=\frac{1-R_0^2}{4\pi^2}\sum_{n=0}^{\infty}\frac{\Gamma^2(n+1)}{(n!)^2}R_0^{2n}=\frac{1-R_0^2}{4\pi^2}\sum_{n=0}^{\infty}R_0^{2n}=\frac{1}{4\pi^2}.$$

Having the series expansion (8.72) of the two-dimensional phase-distribution function, it is not difficult to compute also the correlation function of the phase:

$$B_\varphi(\tau)=m_1\{\varphi(t)\varphi(t+\tau)\}=$$

$$=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Theta_1\Theta_2 w_2(\Theta_1,\Theta_2,\tau,t)\,d\Theta_1 d\Theta_2=$$

$$=\sum_{r=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}A_{rnm}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Theta_1\Theta_2 e^{i(r+n)\Theta_1}e^{i(m-r)\Theta_2}d\Theta_1 d\Theta_2.$$

The variables in the double integral are separable, and the computation of each integral is not difficult. As a result we obtain

$$B_\varphi(\tau)=\sum_{r=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}A_{rmn}\frac{(-1)^{m+n}\cdot 4\pi^2}{(n+r)(m-r)}, \qquad (8.77)$$

$$n\neq -r,\ m\neq r.$$

If there is no signal, then

$$B_\varphi(\tau)=8\pi^2\sum_{r=1}^{\infty}\frac{1}{r^2}A_r(\tau),$$

and, employing (8.76), we find the expansion of the correlation function of a stationary normal random process into a series of powers of R_0

$$B_\varphi(\tau)=2(1-R_0^2)\sum_{r=1}^{\infty}\sum_{n=0}^{\infty}\frac{1}{r^2}\frac{\Gamma^2\left(n+1+\frac{r}{2}\right)}{n!(n+r)!}R_0^{r+2n}= \qquad (8.78)$$

$$=\frac{\pi}{2}R_0(\tau)+\frac{1}{4}R_0^2(\tau)+\frac{7\pi}{12}R_0^3(\tau)+\dots$$

9. Phase Cosine Distribution Functions

In some cases it is necessary to have the statistical characteristics not of the phase $\varphi(t)$, but of $\cos\varphi(t)$. Employing the distribution functions of $\varphi(t)$ obtained in the last chapter, it is not difficult to obtain the distribution functions of $\cos\varphi(t)$. For this it is sufficient to use the relationships which make it

possible to find the laws of distribution for functions of random variables (cf. Sect. 1, Ch. III).

We find first the one-dimensional distribution function of cos φ (t). From (8.38), by means of the replacement of the variable $z = \cos\Theta$, we find the following expression for the one-dimensional probability density of cos φ (when $-\pi \leqslant \Theta \leqslant \pi$ the function $\Theta(z) = \arccos z$ is two-valued):

$$W_1(z)=W_1[\Theta_1(z)]\left|\frac{d\Theta_1}{dz}\right|+W_1[\Theta_2(z)]\left|\frac{d\Theta_2}{dz}\right|, \tag{8.79}$$

and since

$$\left|\frac{d\Theta_1}{dz}\right|=\left|\frac{d\Theta_2}{dz}\right|=\frac{1}{\sqrt{1-z^2}},$$

then*

$$W_1(z)=\frac{1}{\pi\sqrt{1-z^2}}e^{-\frac{s^2}{2}}[1+\sqrt{2\pi}szF(sz)e^{\frac{s^2z^2}{2}}], \quad -1<z<1. \tag{8.80}$$

Figure 66 shows a set of the curves of distribution (8.80) for several values of s. When s = 0 (pure noise) the curve yields the well-known distribution of a harmonic vibration of unit amplitude and random phase [cf. (3.16)]

$$W_1(z)=\frac{1}{\pi\sqrt{1-z^2}}, \quad |z|<1. \tag{8.80'}$$

When $s > 0$ the curve becomes asymmetrical, the probability density for a given s when $z_1 > 0$ is greater than when $z_2 = -z_1$. This is linked to the fact that the probability density of the phase when $|\Theta| < \frac{\pi}{2}$ is greater than when $|\Theta| > \frac{\pi}{2}$, i.e., that positive values of $\cos\Theta$ are more probable than negative ones. Quantitatively this difference is determined from the equality

$$W_1(-z)=W_1(z)-\sqrt{\frac{2}{\pi}}\frac{sz}{\sqrt{1-z^2}}e^{-\frac{s^2}{2}(1-z^2)}, \quad z>0. \tag{8.81}$$

---

* Although the distribution function of the phase and of the phase cosine are denoted by the same letter W, it is necessary in the future to remember, that these are different functions.

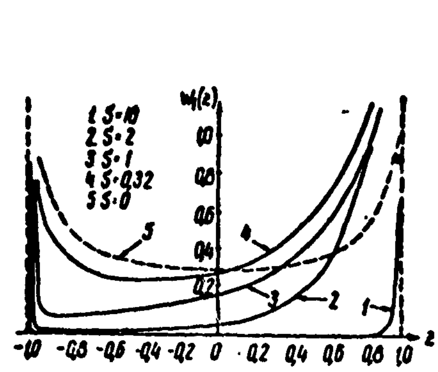


Fig. 66. Distribution function of phase cosine of a normal random process.

Employing expansion (8.49) of the function $W_1 \Theta$ into a Fourier series and bearing in mind, that cos k arc cos z = $T_k(z)$, where $T_k(z)$ is a k-th-order Chebyshev polynomial of the first kind, it is possible to obtain the following expansion of W_1 (z) into a series the Chebyshev polynomials

$$W_1(z)=\frac{1}{\sqrt{1-z^2}}\left[\frac{1}{\pi}+2\sum_{k=1}^{\infty}a_kT_k(z)\right]. \tag{8.82}$$

The coefficients a_k are determined by formula (8.50).

Let us compute the distribution moments of cos φ, making use of expansions (7.17) and (7.18) of the powers of the cosine with respect to cosines of multiple arcs. Taking into account the property of orthogonality of trigonometric functions, we obtain [cf. (8.49)]

$$\begin{aligned} m_{2n}=&\int_{-\pi}^{\pi}\left[\frac{1}{2\pi}+\sum_{r=1}^{\infty}a_r\cos r\Theta\right]\cdot\frac{1}{2^{2n}}\left[\sum_{k=0}^{n-1}2\binom{2n}{k}\cos 2(n-k)\Theta+\right.\\ &\left.+\binom{2n}{n}\right]d\Theta=\frac{1}{2^{2n}}\binom{2n}{n}+\frac{\pi}{2^{2n-1}}\sum_{k=0}^{n-1}a_{2(n-k)}\binom{2n}{k},\end{aligned} \tag{8.83}$$

$$\begin{aligned} m_{2n-1}=&\int_{-\pi}^{\pi}\left[\frac{1}{2\pi}+\sum_{r=1}^{\infty}a_r\cos r\Theta\right]\frac{1}{2^{2n-2}}\left[\sum_{k=0}^{n-1}\binom{2n-1}{k}\cos(2n-\right.\\ &\left.-2k-1)\Theta\right]d\Theta=\frac{\pi}{2^{2n-2}}\sum_{k=0}^{n-1}a_{2n-2k-1}\binom{2n-1}{k}.\end{aligned} \tag{8.84}$$

The first two distribution moments of cos φ are equal to

$$\begin{aligned} m_1=\pi a_1=&\sqrt{\frac{\pi}{2}}\cdot\frac{s}{2}\,{}_1F_1\left(\frac{1}{2},\,2,\,-\frac{s^2}{2}\right)=\\ =&\sqrt{\frac{\pi}{2}}\cdot\frac{s}{2}\left[I_0\left(\frac{s^2}{4}\right)+I_1\left(\frac{s^2}{4}\right)\right]e^{-\frac{s^2}{4}}.\end{aligned} \tag{8.85}$$

$$m_2 = \frac{1}{2} + \frac{\pi a_2}{2} = \frac{1}{2} + \frac{s^2}{8}\,{}_1F_1\left(1, 3, -\frac{s^2}{2}\right). \tag{8.86}$$

We find now the probability, that $\cos\varphi \leqslant z$

$$P\{-1 \leqslant \cos\varphi \leqslant z\} = F_c(z).$$

It is not difficult to express this probability in terms of the integral phase-distribution function $P(\theta)$

$$P\{-1 \leqslant \cos\varphi \leqslant z\} = P\{-\pi \leqslant \varphi \leqslant -\arccos z\} + P\{\arccos z \leqslant \varphi \leqslant \pi\} = 1 - P\{-\arccos z \leqslant \varphi \leqslant \arccos z\},$$

or

$$F_c(z) = 1 - P(\arccos z). \tag{8.87}$$

From (8.87) it follows, that the probability of $1 - F_c(z)$ that $z \leqslant \cos\varphi \leqslant 1$, simply coincides with $P(\arccos z)$, and for computing this probability it is possible to employ the curves and formulas cited in Section 7.

The probability that $\cos\phi$ is positive is equal to

$$1 - F_c(0) = P\left(\frac{\pi}{2}\right) = F(s). \tag{8.88}$$

If there is no signal ($s \equiv 0$), then $P(\arccos z) = \frac{\arccos z}{\pi}$, and then [compare with (3.17)]

$$F_c(z) = 1 - \frac{1}{\pi}\arccos z.$$

To determine the two-dimensional distribution function of a phase cosine it is possible to employ expression (3.69) or (8.72), effecting in them the replacement of variables $z_1 = \cos\theta_1$; $z_2 = \cos\theta_2$.

The inverse functions are two-valued when $-\pi \leqslant \theta_1 \leqslant \pi$, $-\pi \leqslant \theta_2 \leqslant \pi$ therefore to each point of plane (z_1, z_2) there correspond four points of plane (θ_1, θ_2).

$$\theta_{11} = \arccos z_1, \quad \theta_{12} = -\arccos z_1,$$
$$\theta_{21} = -\arccos z_2, \quad \theta_{22} = \arccos z_2.$$

The moduli of the transformation jacobians are, as is not difficult to compute,

equal to

$$\left|\frac{\partial(\Theta_{11},\Theta_{21})}{\partial(z_1,z_2)}\right|=\left|\frac{\partial(\Theta_{11},\Theta_{22})}{\partial(z_1,z_2)}\right|=\left|\frac{\partial(\Theta_{12},\Theta_{21})}{\partial(z_1,z_2)}\right|=\left|\frac{\partial(\Theta_{12},\Theta_{22})}{\partial(z_1,z_2)}\right|=$$
$$=\frac{1}{\sqrt{1-z_1^2}\cdot\sqrt{1-z_2^2}}. \tag{8.89}$$

Let us examine in greater detail the case of the absence of a signal. From expression (8.75), taking into account (8.89) we find

$$W_2(z_1,z_2,\tau)=\frac{1}{\sqrt{1-z_1^2}\cdot\sqrt{1-z_2^2}}[W_2(\Theta_{11},\Theta_{21},\tau)+$$
$$+W_2(\Theta_{11},\Theta_{22},\tau)+W_2(\Theta_{12},\Theta_{21},\tau)+W_2(\Theta_{12},\Theta_{22},\tau)]=$$
$$=\frac{4}{\sqrt{1-z_1^2}\cdot\sqrt{1-z_2^2}}\left\{\frac{1}{4\pi^2}+\sum_{r=1}^{\infty}A_r[\cos(r\arccos z_1)\times\right.$$
$$\times\cos(r\arccos z_2)-\sin(r\arccos z_1)\sin(r\arccos z_2)]+$$
$$+\sum_{r=1}^{\infty}A_r[\cos(r\arccos z_1)\cos(r\arccos z_2)+$$
$$\left.+\sin(r\arccos z_1)\sin(r\arccos z_2)]\right\}=$$
$$=\frac{4}{\sqrt{1-z_1^2}\sqrt{1-z_2^2}}\left[\frac{1}{4\pi^2}+2\sum_{r=1}^{\infty}A_r\cos(r\arccos z_1)\times\right.$$
$$\left.\times\cos(r\arccos z_2)\right],$$

and, introducing the Chebyshev polynomials $T_r(z) = \cos(r \operatorname{arc}\cos z)$, we obtain

$$W_2(z_1,z_2,\tau)=\frac{1}{\pi^2\sqrt{1-z_1^2}\sqrt{1-z_2^2}}\times$$
$$\times\left[1+8\pi^2\sum_{r=1}^{\infty}A_r(\tau)T_r(z_1)T_r(z_2)\right], \tag{8.90}$$
$$-1\leqslant z_1\leqslant 1,\quad -1\leqslant z_2\leqslant 1,$$

where $A_r(\tau)$ is determined according to formula (8.74).

Let us note that series (8.90) represents the expansion of a two-dimensional distribution function of the phase cosine of a stationary normal random process in terms of orthogonal Chebyshev polynomials, which fully corresponds with the general method of resolution, indicated in Sect. 6, Ch. VI, since the one-dimensional distribution function of $\cos\varphi$, equal to $\frac{1}{\pi\sqrt{1-z^2}}$ over the interval of $-1 \leqslant z \leqslant 1$, coincides with the weighting function of polynomials $T_r(z)$.

With the aid of expansion (8.90) it is not difficult to find the correlation function for $\cos\varphi$

$$B(\tau)=m_1\left\{\cos\varphi(t)\cos\varphi(t+\tau)\right\}=\int_{-1}^{+1}\int_{-1}^{+1}z_1z_2W_2(z_1,z_2,\tau)\,dz_1dz_2=$$

$$=2\sum_{r=1}^{\infty}A_r(\tau)\,c_r^2,$$

where

$$c_r=2\int_{-1}^{+1}zT_r(z)\frac{dz}{\sqrt{1-z^2}}=\begin{cases}\pi & r=1,\\ 0 & r>1.\end{cases}$$

In such a manner [cf. (8.74)]

$$B(\tau)=2\pi^2A_1(\tau)=$$

$$=\frac{1}{2(1-R_0^2)}\int_0^{\infty}\int_0^{\infty}z_1z_2I_1\left(\frac{R_0z_1z_2}{1-R_0^2}\right)e^{-\frac{z_1^2+z_2^2}{2(1-R_0^2)}}dz_1dz_2, \tag{8.91}$$

or, taking into account (8.76), we obtain

$$B(\tau)=\frac{1-R_0^2}{2}\sum_{n=0}^{\infty}\frac{\Gamma^2\left(n+\frac{3}{2}\right)}{n!\,(n+1)!}R_0^{2n+1}(\tau)=$$

$$=\frac{\pi R_0(\tau)}{8}\left[1+\frac{R_0^2(\tau)}{8}+\frac{3R_0^4(\tau)}{64}+\ldots+\left[\frac{(2n-1)!!}{2n!!}\right]^2\frac{R_0^{2n}(\tau)}{n+1}+\ldots\right]. \tag{8.92}$$

It is also possible to represent the expression for the correlation function of the phase cosine in another form. For this the double integral of (8.91) should be expressed in terms of hypergeometric functions, which in the case at hand are reduced to full elliptical integrals. Omitting here a presentation of the indicated transformations, we cite only the final result

$$B(\tau)=\frac{1}{2R_0}\left[\mathrm{E}(R_0)-(1-R_0^2)\mathrm{K}(R_0)\right], \tag{8.93}$$

where K and E are full elliptical integrals of the first and second kind, respectively.

The correlation function of $\cos\varphi$ may be computed without preliminary determination of the two-dimensional distribution function, which is particularly important for the case when a signal is present, since in this case determination of the indicated function leads to cumbersome computations. It is clear that (cf. footnote on p. 333)

$$B(\tau)=\int_{-1}^{+1}\int_{-1}^{+1}z_1z_2W_2(z_1,z_2,\tau)\,dz_1dz_2=$$

$$=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cos\Theta_1\cos\Theta_2W_2(\Theta_1,\Theta_2,\tau)\,d\Theta_1\,d\Theta_2. \tag{8.94}$$

where $W_2(\Theta_1, \Theta_2, \tau)$ is the two-dimensional probability density of the phase. Employing (8.72) and changing the order of summation and integration, we find

$$B(\tau)=\sum_{r=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}A_{rnm}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cos\Theta_1\times$$

$$\times\cos\Theta_2 e^{i(r+n)\Theta_1}e^{i(m-r)\Theta_2}\,d\Theta_1 d\Theta_2.$$

In virtue of the orthogonality of the trigonometric functions,

$$\int_{-\pi}^{\pi}\cos\Theta_1 e^{i(n+r)\Theta_1}\,d\Theta_1=\begin{cases}\pi, & n+r=\pm 1\\ 0, & n+r\neq\pm 1,\end{cases}$$

$$\int_{-\pi}^{\pi}\cos\Theta_2 e^{i(m-r)\Theta_2}\,d\Theta_2=\begin{cases}\pi, & m-r=\pm 1\\ 0, & m-r\neq\pm 1.\end{cases}$$

Thus, we obtain the following general expression of the correlation function of $\cos\varphi$ in the form of the series

$$B(\tau)=\pi^2\sum_{r=-\infty}^{\infty}A_{r,\,-r\pm 1,\,r\pm 1}(\tau), \tag{8.95}$$

whose terms are determined by formula (8.71). If there is no signal, then all the terms, with the exception of $A_{1,\,0,\,0}=A_1$, turn to zero and (8.95) turns into (8.91).

10. Statistical Characteristics of the Derivatives of the Envelope and Phase

Let us examine first the more general problem of determining the joint distribution of the envelope, the phase, and their first derivatives for the random process (8.1), assuming that these derivatives exist. In order not to encumber the presentation with analytical computations, we restrict ourselves to the case, when the determined part of the process is a harmonic vibration with the constant amplitude u ($v\equiv 0$).

The starting point for the solution of the indicated problem is a four-dimensional distribution of the envelope and phase of a normal random process

$$\begin{aligned}w_4(r_1, r_2, \Theta_1, \Theta_2, \tau)=\frac{r_1r_2}{(2\pi\sigma^2)^2(1-R_0^2)}\exp\Big\{-\frac{1}{2\sigma^2(1-R_0^2)}\times\\ \times\big[(r_1\cos\Theta_1-u)^2+r_1^2\sin^2\Theta_1+(r_2\cos\Theta_2-u)^2+r_2^2\sin^2\Theta_2-\\ -2R_0\left[(r_1\cos\Theta_1-u)(r_2\cos\Theta_2-u)+r_1r_2\sin\Theta_1\sin\Theta_2\right]\big]\Big\}.\end{aligned} \tag{8.96}$$

We now effect, in distribution function (8.96), the replacement of variables

$$\rho_1=\frac{r_1+r_2}{2},\quad \rho_2=\frac{r_1-r_2}{\tau},$$
$$\psi_1=\frac{\Theta_1+\Theta_2}{2},\quad \psi_2=\frac{\Theta_1-\Theta_2}{\tau}. \tag{8.97}$$

The transformation jacobian of (8.97) is equal to

$$\frac{\partial(\rho_1,\rho_2,\psi_1,\psi_2)}{\partial(r_1,r_2,\Theta_1,\Theta_2)}=\begin{vmatrix}\frac{1}{2} & \frac{1}{2} & 0 & 0\\ \frac{1}{\tau} & -\frac{1}{\tau} & 0 & 0\\ 0 & 0 & \frac{1}{2} & \frac{1}{2}\\ 0 & 0 & \frac{1}{\tau} & -\frac{1}{\tau}\end{vmatrix}=\frac{1}{\tau^2}. \tag{8.98}$$

Therefore

$$\begin{aligned}W_4(\rho_1,\rho_2,\psi_1,\psi_2,\tau)=&\frac{\tau^2\left(\rho_1^2-\frac{\tau^2}{4}\rho_2^2\right)}{(2\pi\sigma^2)^2(1-R_0^2)}\exp\left\{-\frac{1}{2\sigma^2(1-R^2)}\times\right.\\ &\times\left(\left[\left(\rho_1+\frac{\tau}{2}\rho_2\right)\cos\left(\psi_1+\frac{\tau\psi_2}{2}\right)-u\right]^2+\left(\rho_1+\frac{\tau}{2}\rho_2\right)^2\sin^2\left(\psi_1+\right.\right.\\ &\left.+\frac{\tau\psi_2}{2}\right)+\left[\left(\rho_1-\frac{\tau}{2}\rho_2\right)\cos\left(\psi_1-\frac{\tau\psi_2}{2}\right)-u\right]^2+\left(\rho_1-\frac{\tau}{2}\rho_2\right)^2\times\\ &\times\sin^2\left(\psi_1-\frac{\tau}{2}\psi_2\right)-2R_0\left[\left\{\left(\rho_1+\frac{\tau}{2}\rho_2\right)\cos\left(\psi_1+\frac{\tau}{2}\psi_2\right)-u\right\}\times\right.\\ &\times\left\{\left(\rho_1-\frac{\tau}{2}\rho_2\right)\cos\left(\psi_1-\frac{\tau}{2}\psi_2\right)-u\right\}+\\ &\left.\left.\left.+\left(\rho_1+\frac{\tau}{2}\rho_2\right)\left(\rho_1-\frac{\tau}{2}\rho_2\right)\sin\left(\psi_1+\frac{\tau}{2}\psi_2\right)\sin\left(\psi_1-\frac{\tau}{2}\psi_2\right)\right]\right)\right\}.\end{aligned} \tag{8.99}$$

If in (8.99) one is to pass to the limit when $\tau\to 0$, then $\rho_1\to r_1$, $\rho_2\to\frac{dr_1}{d\tau}=r_1'$, $\psi_1\to\Theta_1$, and $\psi_2\to\frac{d\Theta_1}{dt}=\Theta_1'$, and in this manner there will be obtained the desired distribution of the envelope, the phase and of their derivatives in coincident instants of time.

Taking into account, that $R_0(\tau)=1+\frac{\tau^2}{2}R_0''(0)+O(\tau^3)$ and taking the limit, we obtain*

$$W_4(r,r',\Theta,\Theta')=\frac{r^2}{4\pi^2\sigma^4\omega_1^2}e^{-\frac{r'^2+r^2\Theta'^2}{2\sigma^2\omega_1^2}-\frac{r^2-2ur\cos\Theta+u^2}{2\sigma^2}}, \tag{8.100}$$

where there is designated $\omega_1^2=\sqrt{-R_0''(0)}\ (R_0''<0)$.

If the stationary part of the normal process at the input of a linear system is white noise, then it is not difficult to express the magnitude $R_0''(0)=\frac{B_0''(0)}{B_0(0)}$ in terms of the band of the linear system, making use of the appropriate formulas for

---

* The indices "1" of the variables are omitted.

the dispersion of the derivative, which are cited in Sect. 3, Ch. VI. It is then necessary in the indicated formulas to assume $\omega_0 = 0$, since $B_0(\tau)$ results from $B(\tau)$ when $\omega_0 = 0$. Then for the ideal linear system $\omega_1 = \sqrt{-R_0''(0)} = \frac{\Delta}{\sqrt{12}}$, and for a linear system with a gaussian frequency characteristic $\omega_1 = \sqrt{-R_0''(0)} = \frac{\Delta}{\sqrt{2\pi}}$.

From (8.100), integrating along Θ and Θ', we determine the joint distribution function of the envelope and its derivative in coincident instants of time

$$W_2(r, r') = \frac{r^2}{4\pi^2\sigma^4\omega_1^2} e^{-\frac{1}{2\sigma^2}\left\{r^2+u^2+\frac{r'^2}{\omega_1^2}\right\}} \int_{-\pi}^{\pi}\int_{-\infty}^{\infty} e^{-\frac{r^2\Theta'^2}{2\sigma^2\omega_1^2}} e^{-\frac{2ur\cos\Theta}{2\sigma^2}} d\Theta' d\Theta,$$

wherefrom

$$W_2(r, r') = \frac{r}{\sigma^3\omega_1\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}\left\{r^2+u^2+\frac{r'^2}{\omega_1^2}\right\}} I_0\left(\frac{ur}{\sigma^2}\right). \quad (8.101)$$

Comparing (8.101) with (8.3) we find

$$W_2(r, r') = W_1(r)\cdot\frac{1}{\sigma\omega_1\sqrt{2\pi}} e^{-\frac{r'^2}{2\sigma^2\omega_1^2}}, \quad (8.101')$$

i.e., the joint distribution of the envelope and its derivative is equal to the product of the distribution function of the envelope (generalized Rayleigh function) by the distribution function of the derivative, which turns out to be normal with a zero mean and a dispersion of $\sigma^2\omega_1^2$.

Having the expression $W_2(r, r')$, it is not difficult, employing (5.108), to determine the average number of intersections, in a unit of time, of a fixed level $r = r_0$ by the envelope of a normal random process

$$n(x_0) = 2m_1^{(+)} W_1(r_0),$$

where $m_1^{(+)}$ is the average of the positive values of a normally distributed derivative, equal to $m_1^{(+)} = \frac{\sigma\omega_1}{\sqrt{2\pi}}$.

Therefore

$$n(r_0) = 2\cdot\frac{\sigma\omega_1}{\sqrt{2\pi}} W_1(r_0). \quad (8.102)$$

It can be seen from (8.102), that the distribution curves of the envelope, which are shown in Figure 24, represent on a certain scale also the average number of the overshoots of the envelope.

Let us now determine the joint distribution function of the phase and its derivative, integrating (8.100) with respect to r and to r'. Integration with respect to r' is effected immediately, yielding

$$W_2(\Theta,\Theta')=\frac{1}{(2\pi\sigma^2)^{3/2}\omega_1}e^{-\frac{u^2}{2\sigma^2}}\int_0^\infty r^2 e^{-\frac{r^2}{2\sigma^2}\left(1+\frac{\Theta'^2}{\omega_1^2}\right)}e^{-\frac{ur\cos\Theta}{\sigma^2}}dr. \tag{8.103}$$

Finally, integrating with respect to Θ, we find the distribution function of the derivative of the phase of a normal random process

$$\begin{aligned}W_1(\Theta')&=\frac{1}{(2\pi\sigma^2)^{3/2}\omega_1}e^{-\frac{u^2}{2\sigma^2}}\int_{-\pi}^{\pi}\int_0^\infty r^2 e^{-\frac{r^2}{2\sigma^2}\left(1+\frac{\Theta'^2}{\omega_1^2}\right)}e^{-\frac{ur\cos\Theta}{\sigma^2}}dr\,d\Theta=\\&=\frac{1}{\sigma^3\omega_1\sqrt{2\pi}}e^{-\frac{u^2}{2\sigma^2}}\int_0^\infty r^2 I_0\left(\frac{ur}{\sigma^2}\right)e^{-\frac{r^2}{2\sigma^2}\left(1+\frac{\Theta'^2}{\omega_1^2}\right)}dr.\end{aligned} \tag{8.104}$$

The integral in the right part of (8.104) is expressed in terms of a hypergeometric function (cf. Appendix VI). Then $W_1(\Theta')$ may be written in the form of

$$W_1(\Theta')=\frac{e^{-\frac{u^2}{2\sigma^2}}}{\sigma^3\omega_1\sqrt{2\pi}}\cdot\frac{\Gamma\left(\frac{3}{2}\right)\sigma^3\sqrt{2}}{\left(1+\frac{\Theta'^2}{\omega_1^2}\right)^{3/2}}\,{}_1F_1\left(\frac{3}{2},\ 1,\ \frac{u^2}{2\sigma^2\left(1+\frac{\Theta'^2}{\omega_1^2}\right)}\right),$$

or, introducing the designations $s=\frac{u}{\sigma}$, $v=1+\frac{\Theta'^2}{\omega_1^2}$, we obtain

$$W_1(\Theta')=\frac{e^{-\frac{s^2}{2}}}{2\omega_1 v^{3/2}}\,{}_1F_1\left(\frac{3}{2},\ 1,\ \frac{s^2}{2v}\right). \tag{8.105}$$

It can be seen from (8.105), that function $W_1(\Theta')$ is even and the curve corresponding to it is symmetrical with respect to the ordinate axis. A set of the curves of $W_1(\Theta')$ for several values of s is shown in Figure 67.

For a stationary normal random process (u = 0) the distribution function of the derivative of the phase is equal to

$$W_1(\Theta')=\frac{1}{2\omega_1 v^{3/2}}=\frac{1}{2\omega_1}\frac{1}{\left(1+\frac{\Theta'^2}{\omega_1^2}\right)^{3/2}}. \tag{8.106}$$

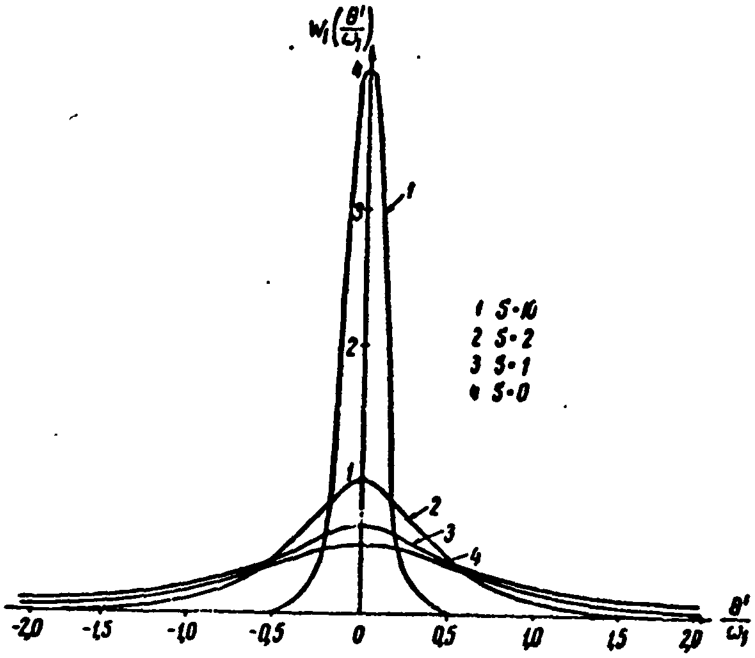


Fig. 67. Distribution function of phase derivative of normal random process.

In this case it is also not difficult to find the probability, that the derivative of the phase will not exceed in absolute magnitude a given frequency Ω.

$$P\{|\Theta'|\leqslant\Omega\}=2\int_0^{\Omega}W_1(\Theta')d\Theta'=\frac{1}{\omega_1}\int_0^{\Omega}\frac{d\Theta'}{\left(1+\frac{\Theta'^2}{\omega_1^2}\right)^{3/2}}= \qquad (8.107)$$
$$=\frac{1}{\sqrt{1+\left(\frac{\omega_1}{\Omega}\right)^2}}.$$

If $\Omega \gg \omega_1$, then

$$P\{|\Theta'|\leqslant\Omega\}\approx 1-\frac{1}{2}\left(\frac{\omega_1}{\Omega}\right)^2. \qquad (8.107')$$

If $u \ll \sigma$, then, resolving the hypergeometric function in (8.105) into a Taylor series and restricting ourselves to two terms of the resolution, we obtain

$$W_1(\Theta')\approx\frac{1}{2\omega_1\left(1+\frac{\Theta'^2}{\omega_1^2}\right)^{3/2}}\left(1+\frac{s^2}{2}\cdot\frac{1}{1+\frac{\Theta'^2}{\omega_1^2}}\right). \qquad (8.108)$$

When $u \gg \sigma$, employing an asymptotic resolution of the hypergeometric function, we find

$$W_1(\Theta') \sim \frac{s}{\omega_1\left(1+\frac{\Theta'^2}{\omega_1^2}\right)^2\sqrt{2\pi}} e^{-\frac{s^2\Theta'^2}{2\omega_1^2\left(1+\frac{\Theta'^2}{\omega_1^2}\right)}}. \tag{8.108'}$$

The magnitude ω_1, as has been indicat d above (cf. Sect. 3, Ch. VI), is proportional to the band of the power spectrum of the initial process. When $\Theta'^2 \ll \omega_1^2$ and u $\gg \sigma$ the distribution of the phase derivative is normal with a dispersion of $\left(\frac{\omega_1}{s}\right)^2$.

The mean value of the phase derivative is, in view of the sy. ry of $W_1(\Theta')$, equal to zero*. In an attempt to compute the dispersion of the phase derivative we are blocked by the divergence of the integral $\int_{-\infty}^{\infty} \Theta'^2 W_1(\Theta') d\Theta'$. In fact, it can be seen from (8.106), for instance, that $W_1(\Theta')$ diminishes when $\Theta' \longrightarrow \infty$ as Θ'^{-3} and, consequently, the integrand function in the indicated integral diminishes as Θ'^{-1}, i.e., too slowly to ensure a convergence of the improper integral. Thus, the dispersion of the phase derivative is unlimited. Distribution (8.105) represents one more example (cf. p. 133) of the distribution of a random function, for which a dispersion does not exist.

As a numerical characteristic of the distribution of a phase derivative, there may be taken the average of its absolute values, i.e.,

$$m_1\{|\dot{\varphi}|\} = \int_{-\infty}^{\infty} |\Theta'| W_1(\Theta') d\Theta' = 2\int_0^{\infty} \Theta' W_1(\Theta') d\Theta'. \tag{8.109}$$

Substituting in (8.109) the expression for $W_1(\Theta')$ from (8.104) and changing the order of integration, we obtain

$$m_1\{|\dot{\varphi}|\} = \frac{2e^{-\frac{u^2}{2\sigma^2}}}{\sigma^2\omega_1\sqrt{2\pi}} \int_0^{\infty} r^2 I_0\left(\frac{ur}{\sigma^2}\right) e^{-\frac{r^2}{2\sigma^2}} \int_0^{\infty} \Theta' e^{-\frac{r^2\Theta'^2}{2\sigma^2\omega_1^2}} d\Theta' dr =$$

$$= \frac{2\omega_1}{\sigma\sqrt{2\pi}} \int_0^{\infty} I_0\left(\frac{ur}{\sigma^2}\right) e^{-\frac{r^2}{2\sigma^2}} dr = \omega_1\, {}_1F_1\left(\frac{1}{2},\ 1,\ -\frac{u^2}{2\sigma^2}\right),$$

or, expressing the hypergeometric function in terms of a Bessel function (cf. Appendix VI), we find

---

* Cf. Footnote, p. 301. Consequently, here we are speaking of the mean value of the deviation of the frequency from ω_0.

$$m_1\{|\dot{\varphi}|\}=\omega_1 e^{-\frac{u^2}{4\sigma^2}} I_0\left(\frac{u^2}{4\sigma^2}\right). \tag{8.110}$$

If the determined part of the process is lacking (u = 0), then

$$m_1\{|\dot{\varphi}|\}=\omega_1. \tag{8.111}$$

11. Correlation Function and Power Spectrum of Process at the Output of an FM Discriminator

To conclude the present chapter let us find the correlation function and power spectrum of the phase derivative of a normal stationary random process.

This problem assumes still more significance with the development of frequency-modulation engineering. As is well known, a receiver designed for the reception of frequency-modulated oscillations has, following the amplifier, two nonlinear elements: a limiter and an FM discriminator. If the effective width of the power spectrum of a normal stationary random process acting on the rf-amplifier input is much greater than its pass bandwidth, then the normal random process ξ (t) at the output of the rf-amplifier will be a narrow-band one, and may in accordance with (6.39) be represented in the form:

$$\xi(t)=A(t)\cos\omega_0 t+C(t)\sin\omega_0 t=$$
$$=\sqrt{A^2(t)+C^2(t)}\cos\left(\omega_0 t-\operatorname{arctg}\frac{C(t)}{A(t)}\right).$$

If, besides that, the power spectrum of ξ (t) is symmetrical with respect to the central frequency ω_0, then A(t) and C(t) are uncorrelated, and their correlation functions B(τ) are equal to each other. To simplify the mathematical calculations, it may be assumed that, after limiting, the process is equal to*

$$\xi_1(t)=\cos\left[\omega_0 t-\operatorname{arctg}\frac{C(t)}{A(t)}\right].$$

Then the random process at the output of the FM discriminator coincides with the phase derivative of a normal random process

$$\Omega(t)=-\frac{d}{dt}\operatorname{arctg}\frac{C(t)}{A(t)}=\frac{C(t)\cdot A'(t)-A(t)\cdot C'(t)}{A^2(t)+C^2(t)}. \tag{8.112}$$

---

* A solution of the problem with more general assumpti , concerning the limits characteristics is cited in [14] .

The correlation function of random process Ω (t) is equal to

$$B_{\Omega}(\tau)=m_1\{\Omega(t)\Omega(t+\tau)\}=$$
$$=m_1\left\{\frac{C(t)A'(t)-A(t)C'(t)}{A^2(t)+C^2(t)}\cdot\frac{C(t+\tau)A'(t+\tau)-A(t+\tau)C'(t+\tau)}{A^2(t+\tau)+C^2(t+\tau)}\right\}. \tag{8.113}$$

It can be seen from (8.113), that in order to determine $B_{\Omega}(\tau)$, it is necessary to have an eight-order distribution function of the random variables A(t), A'(t), A(t + τ), A'(t + τ), C(t), C'(t), C(t + τ), C'(t + τ).

In virtue of the fact that the random variables A(t) and C(t) are uncorrelated, the indicated distribution function is equal to the product of two normal distribution functions of the fourth order, for each of which the defining determinant has the form of (5.103).

For subsequent computations it is convenient to employ an integral representation of these distribution functions in terms of the four-dimensional characteristic functions corresponding to them (cf. 3.35').

$$w_4(x_1, x_1', x_2, x_2')=$$
$$=\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\frac{\sigma^2}{2}\sum_{l=1}^{4}\sum_{k=1}^{4}r_{lk}v_lv_k}e^{-i(x_1v_1+\ldots+x_2'v_4)}dv_1\ldots dv_4, \tag{8.114}$$

$$w_4(y_1, y_1', y_2, y_2')=$$
$$=\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\frac{\sigma^2}{2}\sum_{l=1}^{4}\sum_{k=1}^{4}r_{lk}u_lu_k}e^{-i(y_1u_1+\ldots+y_2'u_4)}du_1\ldots du_4, \tag{8.115}$$

where r_{lk} coincides with the elements of determinant (5.103), and σ^2 = B(0).

Taking into account (8.114) and (8.115), we find

$$B_{\Omega}(\tau)=\underbrace{\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}}_{8\text{ times}}\frac{y_1x_1'-x_1y_1'}{x_1^2+y_1^2}\times$$
$$\times\frac{y_2x_2'-x_2y_2'}{x_2^2+y_2^2}w_4(x_1,\ldots x_2')w_4(y_1,\ldots y_2')dx_1\ldots dy_2'= \tag{8.116}$$
$$=\frac{1}{(2\pi)^8}\underbrace{\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}}_{16\text{ times}}\frac{y_1x_1'-x_1y_1'}{x_1^2+y_1^2}\cdot\frac{y_2x_2'-x_2y_2'}{x_2^2+y_2^2}e^{-\frac{\sigma^2}{2}\sum_{l=1}^{4}\sum_{k=1}^{4}r_{lk}(v_lv_k+u_lu_k)}\times$$
$$\times e^{-i(x_1v_1+\ldots+y_2'u_4)}dx_1\ldots dy_2'dv_1\ldots du_4.$$

The integration with respect to x_1, x_1', y_1, y_1', and x_2, x_2', y_2, y_2' ...

in (8.116) is separable and reduces to the computation of four fourfold integrals of the same type.

Let us examine the integral

$$K_1=\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{y_1x_1'}{x_1^2+y_1^2}e^{-i(x_1v_1+x_1'v_2+y_1u_1+y_1'u_2)}dx_1dx_1'dy_1dy_1'.$$

Employing the delta-function (cf. Appendix IV), it is possible to represent this integral in the form:

$$K_1=\frac{i}{(2\pi)^2}\delta'(v_2)\delta(u_2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{y_1}{x_1^2+y_1^2}e^{-i(x_1v_1+y_1u_1)}dx_1dy_1.$$

Transformation should now be made from the variables x_1 and y_1 to the polar coordinates ρ and φ.

$$x_1=\rho\cos\varphi,\ y_1=\rho\sin\varphi;$$

then

$$K_1=\frac{i}{(2\pi)^2}\delta'(v_2)\delta(u_2)\int_0^{\infty}\int_0^{2\pi}\sin\varphi e^{i\rho(v_1\cos\varphi+u_1\sin\varphi)}d\varphi d\rho.$$

But

$$\frac{i}{2\pi}\int_0^{2\pi}\sin\varphi e^{i\rho(v_1\cos\varphi+u_1\sin\varphi)}d\varphi=$$

$$=\frac{1}{4\pi}\int_0^{2\pi}(e^{i\varphi}-e^{-i\varphi})e^{i\rho\sqrt{v_1^2+u_1^2}\sin(\varphi+\psi)}d\varphi=$$

$$=\frac{1}{2}\left\{e^{-i\psi}\cdot\frac{1}{2\pi}\int_{\psi}^{2\pi+\psi}e^{i\theta+i\rho\sqrt{v_1^2+u_1^2}\sin\theta}d\theta-\right.$$

$$\left.-e^{i\psi}\cdot\frac{1}{2\pi}\int_{\psi}^{2\pi+\psi}e^{-i\theta+i\rho\sqrt{v_1^2+u_1^2}\sin\theta}d\theta\right\}=$$

$$=J_1(\rho\sqrt{v_1^2+u_1^2})\frac{e^{i\psi}+e^{-i\psi}}{2}=J_1(\rho\sqrt{v_1^2+u_1^2})\cos\psi,$$

where

$$\cos\psi=\frac{u_1}{\sqrt{v_1^2+u_1^2}}.$$

Thus

$$K_1=\frac{1}{2\pi}\cdot\frac{u_1}{\sqrt{v_1^2+u_1^2}}\delta'(v_2)\delta(u_2)\int_0^{\infty}J_1(\rho\sqrt{v_1^2+u_1^2})d\rho,$$

and since

$$\int_0^\infty J_1(\rho\sqrt{v_1^2+u_1^2})\,d\rho = \frac{1}{\sqrt{v_1^2+u_1^2}},$$

then

$$K_1 = \frac{1}{2\pi}\cdot\frac{u_1}{v_1^2+u_1^2}\,\delta'(v_2)\,\delta(u_2). \tag{8.117}$$

Analogously

$$K_2 = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{x_1y_1'}{x_1^2+y_1^2}\times$$

$$\times e^{-i(x_1v_1+x_1'v_2+y_1u_1+y_1'u_2)}\,dx_1dx_1'dy_1dy_1' = \tag{8.118}$$

$$= \frac{1}{2\pi}\cdot\frac{v_1}{v_1^2+u_1^2}\,\delta(v_2)\,\delta'(u_2).$$

$$K_3 = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{y_2x_2'}{x_2^2+y_2^2}\times$$

$$\times e^{-i(x_2v_3+x_2'v_4+y_2u_3+y_2'u_4)}\,dx_2dx_2'dy_2dy_2' = \tag{8.119}$$

$$= \frac{1}{2\pi}\,\frac{u_3}{v_3^2+u_3^2}\,\delta'(v_4)\,\delta(u_4).$$

$$K_4 = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{x_2y_1'}{x_2^2+y_2^2}\times$$

$$\times e^{-i(x_2v_3+x_2'v_4+y_2u_3+y_2'u_4)}\,dx_2dx_2'dy_2dy_2' = \tag{8.120}$$

$$= \frac{1}{2\pi}\,\frac{v_3}{v_3^2+u_3^2}\,\delta(v_4)\,\delta'(u_4).$$

Substituting (8.117) - (8.120) into (8.116) and taking into account the filtering property of the delta-function and of its derivative, we obtain

$$B_y(\tau) = \frac{1}{(2\pi)^2}\underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{8\ \text{times}}\frac{1}{v_1^2+u_1^2}\times$$

$$\times\frac{1}{v_3^2+u_3^2}\,[u_1\delta'(v_2)\,\delta(u_2)+v_1\delta(v_2)\,\delta'(u_2)]\times$$

$$\times[u_3\delta'(v_4)\,\delta(u_4)+v_3\delta(v_4)\,\delta'(u_4)]\times$$

$$\times e^{-\frac{\sigma^2}{2}\sum_{l=1}^{4}\sum_{k=1}^{4}r_{lk}(v_lv_k+u_lu_k)}\,dv_1\ldots du_4 =$$

$$=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sigma^2 R''(v_1v_3+u_1u_3)+\sigma^4 R'^2(u_1v_3-v_1u_3)^2}{(v_1^2+u_1^2)(v_3^2+u_3^2)}\times$$

$$\times\, e^{-\frac{\sigma^2}{2}(v_1^2+u_1^2+v_3^2+u_3^2+2Rv_1v_3+2Ru_1u_3)}\,dv_1dv_3du_1du_3.$$

Subsequent computations of the multiple integral are simplified, if use is made of the substitution of the variables of integration:

$$v_1=\frac{r}{\sigma}\cos(\alpha+\beta),\ u_1=\frac{r}{\sigma}\sin(\alpha+\beta).$$

$$v_3=\frac{\rho}{\sigma}\cos\beta,\ u_3=\frac{\rho}{\sigma}\sin\beta.$$

Then
$$B_\Omega(\tau)=\frac{1}{4\pi^2}\int_0^{\infty}\int_0^{\infty}\int_0^{2\pi}\int_0^{2\pi}(R''\cos\alpha+R'^2 r\rho\sin^2\alpha)\times$$

$$\times\, e^{-\frac{1}{2}(r^2+\rho^2+2Rr\rho\cos\alpha)}\,d\beta\, d\alpha\, dr\, d\rho=$$

$$=\frac{1}{2\pi}\int_0^{\infty}\int_0^{\infty}e^{-\frac{1}{2}(r^2+\rho^2)}\left(R''\int_0^{2\pi}\cos\alpha\, e^{-Rr\rho\cos\alpha}\,d\alpha+\right.$$

$$\left.+R'^2 r\rho\int_0^{2\pi}\sin^2\alpha\, e^{-Rr\rho\cos\alpha}\,d\alpha\right)dr\,d\rho.$$

Taking into account that $\frac{1}{2\pi}\int_0^{\infty}\cos n\alpha\, e^{-a\cos\alpha}d\alpha=(-1)^n I_n(a)$ and employing formulas (3) and (7) of Appendix VI, it is not difficult to carry the integration out to the end and to obtain the desired expression of the correlation function of the random process at the output of an FM discriminator

$$B_\Omega(\tau)=\frac{1}{2}[R'^2(\tau)-R''(\tau)R(\tau)]\left[1+\frac{R^2(\tau)}{2}+\frac{R^4(\tau)}{3}+\dots\right]=$$
$$=-\frac{1}{2}[R'^2(\tau)-R''(\tau)R(\tau)]\frac{\ln[1-R^2(\tau)]}{R^2(\tau)}. \qquad (8.121)^*$$

The power spectrum of a random process at the output of an FM discriminator is obtained by means of a Fourier transformation effected on $B_\Omega(\tau)$.

Let the random process at the input of the rf-amplifier be "white noise", and the frequency characteristic of the rf-amplifier be a gaussian curve. Then taking

---

* Since $R(0) = 1$, then from (8.121) it follows that the magnitude $B_\Omega(\tau)$ becomes unlimited when $\tau \to 0$. However, taking into account the finite width of the discriminator-filter pass band, the correlation function of the process at the output of a FM receiver will have a finite value when $\tau = 0$.

into account (6.19) and (6.46), we find an expression for the correlation coefficient playing a role in (8.121)

$$R(\tau)=e^{-\frac{\beta^2\tau^2}{4}}, \tag{8.122}$$

wherefrom

$$R'(\tau)=-\frac{\beta^2\tau}{2}e^{-\frac{\beta^2\tau^2}{4}},\quad R''(\tau)=-\frac{\beta^2}{2}\left(1-\frac{\beta^2\tau^2}{2}\right)e^{-\frac{\beta^2\tau^2}{4}}. \tag{8.123}$$

In (8.122) and (8.123) the parameter β is linked with the power band width of the rf-amplifier by the relationship $\Delta = \beta\sqrt{\pi}$. Substituting (8.122) and (8.123) into (8.121), we find

$$B_2(\tau)=-\frac{1}{2}\left[\frac{\beta^4\tau^2}{4}+\frac{\beta^2}{2}\left(1-\frac{\beta^2\tau^2}{2}\right)\right]\ln\left(1-e^{-\frac{\beta^2\tau^2}{2}}\right)= -\frac{\beta^2}{4}\ln\left(1-e^{-\frac{\beta^2\tau^2}{2}}\right). \tag{8.124}$$

The power spectrum is, in accordance with (5.44) equal to

$$F_2(\omega)=-\beta^2\int_0^\infty \cos\omega\tau\ln\left(1-e^{-\frac{\beta^2\tau^2}{2}}\right)d\tau.$$

Expanding the logarithm into a series and integrating by terms, we obtain

$$F_2(\omega)=\beta^2\sum_{n=1}^{\infty}\frac{1}{n}\int_0^\infty e^{-\frac{n\beta^2\tau^2}{2}}\cos\omega\tau d\tau = \beta\sqrt{\frac{\pi}{2}}\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}e^{-\frac{\omega^2}{2n\beta^2}}. \tag{8.125}$$

The intensity of the power spectrum when $\omega = 0$ is equal to

$$F_2(0)=\beta\sqrt{\frac{\pi}{2}}\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}=\beta\sqrt{\frac{\pi}{2}}\zeta(3/2),$$

where $\zeta(x)$ is a Riemann zeta-function. Bearing in mind, that $\zeta(3/2) = 2.612$*, we find

$$F_2(0)=1{,}86\beta\sqrt{\pi}=1{,}86\Delta.$$

When $\omega \gg \beta$

$$F_2(\omega)\sim\frac{\pi\beta^2}{\omega}=\frac{\Delta^2}{\omega}.$$

---

* Cf. e.g., Ye. Yanke and F. Emde. Tablitsy funktsiy (Tables of functions), Gostekhizdat, 1948, p. 372.

Figure 68 shows the curve of power spectrum (8.125).

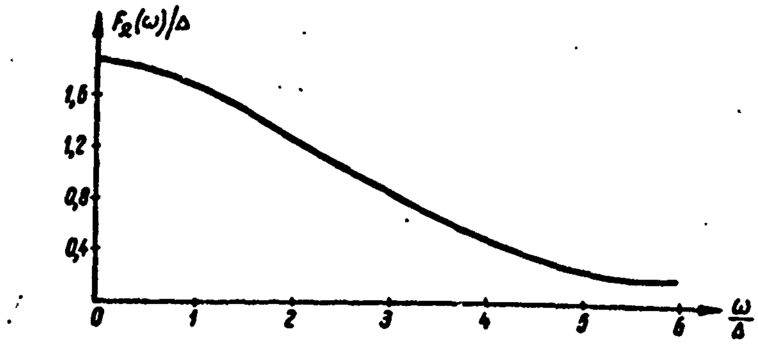


Fig. 68. Power spectrum of process at output of FM discriminator.

Literature

1. V. I. Bunimovich. Fluktuatsionnyye protsessy v radiopriyemnykh ustroistvakh (Fluctuation processes in radio receivers). Moskva. "Sovetskoye Radio" Pub. Hse., 1951.

2. D. Middleton. Some General Results in the Theory of Noise Through Nonlinear Devices. Quart. Appl. Math. 5, No. 4, Jan. 1948.

3. D. Middleton. Statistical Criteria for the Detection of Pulsed Carriers in Noise. I u II. Journ. Appl. Phys., 24, No. 4, 1953.

4. S. O. Rice. Mathematical Analysis of Random Noise. BSTJ, 23, No. 3, July 1944; 24, No. 1, Jan. 1945. (A translation [into Russian] is available in the collection "Teoriya peredachi elektrechneskikh signalov pri nalichii pomekh" (Theory of the transmission of electrical signals in the presence of interference), For. Lit. Pub. Hse., 1953.)

5. Porogovyye signaly (Threshold Signals). Translation edited by A. P. Sivers, "Sovetskoye Radio", 1952.

6. W. W. Peterson, T. G. Birdsall, W. C. Fox. The Theory of Signal Detectability Trans. IRE, PGIT-4, Sept. 1954.

7. F. M. Vudvord (Woodward). Teoriya veroyatnostey i teoriya informatsii s primenneniyami v radiolokatsii (Probability theory and information theory with applications in radiolocation). Translation edited by G. S. Gorelik. Moskva,

"Sovetskoye radio" Pub. Hse., 1955.

8. V. I. Tikhonov, The Influence of Electrical Fluctuations on a Detector (the envelope method). Izvestiya ANSSSR, Otdeleniye nauk (News, USSR Academy of Sciences, Division of Technical Sciences), No. 10, 1955.

9. W. C. Hoffman. The Joint Distribution of n Successive Outputs of a Linear Detector. Journ. Appl. Phys. 25, No. 8, 1954.

10 D. Middleton. On the Theory of Random Noise, I and II. Journ. Appl. Phys. 22, No. 9, 1951.

11. K. A. Norton, E. L. Shultz, H. Yarhough. The Probability Distribution of the Phase of the Resultant Vector Sum of a Constant Plus a Rayleigh Distributed Vector. Journ. Appl. Phys., 23, No. 1, Jan. 1952.

12. A. M. Bonch-Bruyevich, V. I. Shirokov. Certain Questions of Phase Measurement. ZhTF (Zhurnal Tekhnicheskoy Fiziki - Journal of Technical Physics), 1955, V. 25, No. 10.

13. S. O. Rice. Statistical Properties of a Sine Wave Plus Random Noise. BSTJ, 27, Jan. 1948.

14. D. Middleton. The Spectrum of Frequency-Modulated Waves after Reception in Random Noise. I. II Quart. App. Math. July 1949, Apr. 1950.

Chapter IX

PASSAGE OF NORMAL RANDOM PROCESS THROUGH STANDARD RADIO-EQUIPMENT SECTIONS

1. Statement of the Question and the Method of Solving it

It has already been noted in Sect. 1, Ch. VI that a characteristic feature of many stages in the operation of radio equipment is the transformation of electrical signals, which generally speaking are random processes, in a standard section consisting of three consecutive elements: the input linear system, the nonlinear (non-inertial) element, and the output linear system. (Fig. 43).

If a normal random process is acting on the input of a standard section, then finding the power spectrum of the process at its output presents in principle no difficulties. Past the input linear system the process remains normal, and the spectrum is deformed in accordance with the shape of the frequency characteristic of this linear system [cf. (6.4)]. As a result of nonlinear transformation the distribution functions of the process cease to be normal, but the spectrum of the transformed process can still be determined through the use of one of the methods set forth in detail in Chapter VII. Thereafter it is sufficient to take into account the selective action of the output linear system, employing formula (6.4).

However, in many cases a knowledge of the power spectrum of the process at the output of a standard section is insufficient, and it is necessary to know such finer characteristics of random processes as the distribution functions. The determination of distribution functions is tied in with considerable difficulties, of both a theoretical and a computational nature, since for this it is necessary to solve the problem of the transformation of the distribution function of random process in a linear system upon whose input there acts a process which is not normal (cf. Sect. 8, Chapter VI).

In those cases where the process at the input of a standard section is not normal, this difficulty appears already in the first stage of the investigation. It

may constitute an obstacle not only to the solution of the problem of determining the distribution functions, but even to that of the problem of finding the power spectrum of a process past the nonlinear element, since for this it is necessary to know the second distribution function of a random process at the input of the non-linear element (i.e., at the output of the preceding linear system).

There exists a limited number of precise solutions to the problem of determining the distribution function of a process at the output of a standard section, which have been obtained with some special assumptions as to the nonlinearity characteristic and the statistical properties of the random process at the input.

An approximate method of determining the one-dimensional distribution function consists in computing a certain number of distribution moments. This method permits generalization [4], but even in its simplest form, as it was set forth in Sect. 8, Chapter VI, its practical application is tied in with cumbersome calculations.

In the present chapter there is examined a precise solution to the problem of determining the one-dimensional distribution function of a process at the output of a standard section, with two fundamental restrictions:

1) the random process at the input constitutes the sum of a determined signal $S(t)$ and a stationary normal random process with a uniform spectrum ("white noise"), and with an intensity (average power per unit of band) equal to σ^2;

2) the characteristic of the non-linear element is quadratic: $y = x^2$ (the constant multiplier is omitted).*

The problem under consideration obviously coincides with the following problem, of important significance in many radio-engineering applications (location, communications, etc.).

The input of an rf-amplifier is acted upon by a determined signal $S(t)$ and by fluctuation noises. The signal is subjected with the noises to square-law detectors

---

* A solution of the problem for more general assumptions, where the multiplier serves as the nonlinear element, is cited in [7].

and to the subsequent filtration. What is the distribution function of the signal and the noises at the output of the filter?

In the following exposition, the terminology connected with this special problem is preserved for the sake of greater definiteness. Two cases are considered: a) a wide-band rf-amplifier and b) a narrow-band rf-amplifier (the width of the frequency-characteristic band is much less than the central frequency). In the first case the process after nonlinear transformation is assumed equal to the square of the random process at the output of the rf-amplifier. In the second case the process after nonlinear transformation is assumed equal to the square of the envelope of the random process at the output of the rf-amplifier (i.e., the high-frequency component of the process is discarded).

We shall show that the processes at the input and output of the standard section in question are linked by an integral relationship. Let the linear systems of the standard section be characterized by their pulse transfer functions: the rf-amplifier by the function $h_1(\tau)$ and the filter by the function $h_2(\tau)$. The link between the pulse transfer function and the square of the frequency characteristic of a linear system is provided by the formulas in Sect. 2, Ch. VI).

Let us designate white normal noise by $\xi(t)$. Since the signal and the noise pass through a linear system independently, the process at the output of the rf-amplifier will be a sum of two items: the determined $S_1(t)$ and the stationary normal random process $\xi_1(t)$, each of which may be represented by Duhamel's integral (cf. 6.3)

$$S_1(t)=\int_{-\infty}^{\infty} h_1(\tau)S(t-\tau)d\tau, \tag{9.1}$$

$$\xi_1(t)=\int_{-\infty}^{\infty} h_1(\tau)\xi(t-\tau)d\tau. \tag{9.2}$$

From (9.1) and (9.2) there follows the possibility of representing the square of the random process at the output of the rf-amplifier in the form of

$$[S_1(t)+\xi_1(t)]^2=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h_1(u)[S(t-u)+\xi(t-u)]\,h_1(v)[S(t-v)+$$
$$+\xi(t-v)]\,du\,dv. \tag{9.3}$$

If $\xi_2(t)$ is the process after filtration (i.e., at the output of the standard section), then

$$\xi_2(t)=\int_{-\infty}^{\infty}h_2(\tau)[S_1(t-\tau)+\xi_1(t-\tau)]^2\,d\tau, \tag{9.4}$$

or, substituting into (9.4) the expression (9.3) and replacing the variables of integration u and v by u-τ and v - τ, we obtain

$$\xi_2(t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K(u,v)[S(t-u)+\xi(t-u)][S(t-v)+$$
$$+\xi(t-v)]\,du\,dv, \tag{9.5}$$

where

$$K(u,v)=\int_{-\infty}^{\infty}h_1(u-\tau)\,h_2(\tau)\,h_1(v-\tau)\,d\tau. \tag{9.6}$$

Expression (9.5) represents the desired integral transformation, by which the process at the output of the standard section is linked to the process at its input. We shall call the function K(u,v), which depends only on the characteristics of the rf-amplifier and the filter, the nucleus of this transformation*.

In problems dealing with the envelope of the process at the output of a standard section, there is placed under the sign of integral (9.4) not the square of the random process at the output of the rf-amplifier, but the square of its envelope. Here it is useful to employ the concept of the narrow-band normal random process in the form of a sum [cf. (6.39)]

$$\xi_1(t)+S_1(t)=[\xi_A(t)+S_A(t)]\cos\omega_0 t+$$
$$+[\xi_c(t)+S_c(t)]\sin\omega_0 t. \tag{9.7}$$

Thus the solution of the problem at hand has been reduced to a determination of the statistical characteristics of integral (9.5), for which there will be necessary a more detailed study of the properties of such integral transformations.

---

* In taking into account the physical feasibility of the rf-amplifier and the filter, it should be assumed that $h_1(t) = 0$, $h_2(t) = 0$ when $t < 0$.

Since consideration has been given above to the nonlinear transformations only of the envelope of a narrow-band normal random process(Sec.4Chapter VIII)and not to the nonlinear transformations of wide-band normal random processes, therefore #2, which contains a detailed computation of the first two distribution functions of the square of a wide-band normal random process, is prerequisite to a solution of the principal problem to which the present chapter is devoted. In a certain measure this section may serve as a standard example in the determination of the distribution functions of random processes undergoing nonlinear transformations.

2. Distribution Functions of the Square of a Normal Random Process

Let us find the two-dimensional distribution function of the square of a normal random process, having employed the general formulas for the replacement of variables in distribution functions undergoing transformation (cf. Sect. 1, Ch. III):

$$\eta_1 = \xi_1^2, \quad \eta_2 = \xi_2^2,$$

where

$$\xi_1 = \xi(t), \quad \xi_2 = \xi(t+\tau).$$

We designate by $w_2(x_1, x_2, \tau, t)$ and $W_2(y_1, y_2, \tau, t)$ the two-dimensional distribution functions of a normal random process and, respectively, of its square. Since the function inverse to $y = f(x) = x^2$ is two-valued, to each point with the coordinates $y_1 > 0$, $y_2 > 0$ there will correspond four points in plane (x_1, x_2):

$$\begin{aligned} x_{11} &= \sqrt{y_1}, \quad x_{12} = -\sqrt{y_1}, \\ x_{21} &= \sqrt{y_2}, \quad x_{22} = -\sqrt{y_2}. \end{aligned} \tag{9.8}$$

Then in accordance with (3.7)

$$\begin{aligned} W_2(y_1, y_2, \tau, t) &= w_2(x_{11}, x_{21}, \tau, t)\left|\frac{\partial(x_{11}, x_{21})}{\partial(y_1, y_2)}\right| + \\ &+ w_2(x_{11}, x_{22}, \tau, t)\left|\frac{\partial(x_{11}, x_{22})}{\partial(y_1, y_2)}\right| + w_2(x_{12}, x_{21}, \tau, t)\left|\frac{\partial(x_{12}, x_{21})}{\partial(y_1, y_2)}\right| + \\ &+ w_2(x_{12}, x_{22}, \tau, t)\left|\frac{\partial(x_{12}, x_{22})}{\partial(y_1, y_2)}\right|. \end{aligned} \tag{9.9}$$

The absolute values of all four jacobians in (9.9) do not differ from each other and are equal to $\frac{1}{4\sqrt{y_1 y_2}}$.

Substituting the expression (7.1) of the two-dimensional function of a normal random process into (9.9) and taking into account (9.8), we find

$$W_2(y_1, y_2, \tau, t) = \frac{1}{4\sqrt{y_1 y_2}} \cdot \frac{1}{2\pi\sigma^2\sqrt{1-R^2}} e^{-\frac{a_1^2+a_2^2-2Ra_1a_2}{2\sigma^2(1-R^2)}} \times$$

$$\times e^{-\frac{y_1+y_2}{2\sigma^2(1-R^2)}} \left(e^{\frac{(a_1-Ra_2)\sqrt{y_1}+(a_2-Ra_1)\sqrt{y_2}+R\sqrt{y_1y_2}}{\sigma^2(1-R^2)}} + \right.$$

$$+ e^{\frac{(a_1-Ra_2)\sqrt{y_1}-(a_2-Ra_1)\sqrt{y_2}-R\sqrt{y_1y_2}}{\sigma^2(1-R^2)}} +$$

$$+ e^{-\frac{(a_1-Ra_2)\sqrt{y_1}+(a_2-Ra_1)\sqrt{y_2}-R\sqrt{y_1y_2}}{\sigma^2(1-R^2)}} +$$

$$\left. + e^{-\frac{(a_1-Ra_2)\sqrt{y_1}-(a_2-Ra_1)\sqrt{y_2}+R\sqrt{y_1y_2}}{\sigma^2(1-R^2)}} \right) =$$

$$= \frac{1}{2\sqrt{y_1y_2}} \cdot \frac{1}{2\pi\sigma^2\sqrt{1-R^2}} e^{-\frac{a_1^2+a_2^2-2Ra_1a_2}{2\sigma^2(1-R^2)}} e^{-\frac{y_1+y_2}{2\sigma^2(1-R^2)}} \times$$

$$\times \left\{ e^{\frac{R\sqrt{y_1y_2}}{\sigma^2(1-R^2)}} \operatorname{ch}\left[\frac{(a_1-Ra_2)\sqrt{y_1}+(a_2-Ra_1)\sqrt{y_2}}{\sigma^2(1-R^2)}\right] + \right.$$

$$\left. + e^{-\frac{R\sqrt{y_1y_2}}{\sigma^2(1-R^2)}} \operatorname{ch}\left[\frac{(a_1-Ra_2)\sqrt{y_1}-(a_2-Ra_1)\sqrt{y_2}}{\sigma^2(1-R^2)}\right] \right\}.$$

Transforming the hyperbolic cosines of the sum and difference and grouping the terms with cosines and sines, we find the desired expression for the two-dimensional distribution function of the square of a normal random process:

$$W_2(y_1, y_2, \tau, t) = \frac{1}{2\pi\sigma^2\sqrt{y_1y_2(1-R^2)}} e^{-\frac{a_1^2+a_2^2-2Ra_1a_2}{2\sigma^2(1-R^2)}} e^{-\frac{y_1+y_2}{2\sigma^2(1-R^2)}} \times$$

$$\times \left\{ \operatorname{ch}\left[\frac{R\sqrt{y_1 y_2}}{\sigma^2(1-R^2)}\right] \operatorname{ch}\left[\frac{\sqrt{y_1}(a_1-Ra_2)}{\sigma^2(1-R^2)}\right] \operatorname{ch}\left[\frac{\sqrt{y_2}(a_2-Ra_1)}{\sigma^2(1-R^2)}\right] + \right. \quad (9.10)$$

$$\left. + \operatorname{sh}\left[\frac{R\sqrt{y_1 y_2}}{\sigma^2(1-R^2)}\right] \operatorname{sh}\left[\frac{\sqrt{y_1}(a_1-Ra_2)}{\sigma^2(1-R^2)}\right] \operatorname{sh}\left[\frac{\sqrt{y_2}(a_2-Ra_1)}{\sigma^2(1-R^2)}\right] \right\},$$

$$y_1 > 0, \quad y_2 > 0.$$

If the determined part is missing ($a_1 = a_2 = 0$), then from (9.10) we obtain the two-dimensional distribution function of the square of a stationary normal random process

$$W_2(y_1, y_2, \tau) = \frac{1}{2\pi\sigma^2\sqrt{y_1y_2(1-R^2)}} e^{-\frac{y_1+y_2}{2\sigma^2(1-R^2)}} \operatorname{ch}\left[\frac{R\sqrt{y_1y_2}}{\sigma^2(1-R^2)}\right], \quad (9.11)$$

$$y_1 > 0, \quad y_2 > 0.$$

It is not difficult to obtain a one-dimensional distribution function from (9.10),

if $\tau \longrightarrow \infty$ ($R \rightarrow 0$); then

$$W_1(y,t) = \frac{1}{\sqrt{2\pi\sigma^2 y}} e^{-\frac{y+a^2}{2\sigma^2}} \operatorname{ch}\left(\frac{a\sqrt{y}}{\sigma^2}\right), \quad y > 0. \tag{9.12}$$

When a = 0, we obtain the probability density of the square of a random variable distributed according to the normal law

$$W_1(y) = \frac{1}{\sqrt{2\pi\sigma^2 y}} e^{-\frac{y}{2\sigma^2}}, \quad y > 0, \tag{9.13}$$

which does not differ from (3.14).

If $a \gg \sigma$, then in (9.12) the hyperbolic cosine may be replaced by its asymptotic approximation

$$\operatorname{ch}\left(\frac{a\sqrt{y}}{\sigma^2}\right) \sim \frac{1}{2} e^{\frac{a\sqrt{y}}{\sigma^2}}.$$

Then distribution function (9.12) may be rewritten in the form of

$$W_1(y, t) \sim \frac{1}{2\sqrt{2\pi\sigma^2 y}} e^{-\frac{(\sqrt{y}-a)^2}{2\sigma^2}}. \tag{9.14}$$

Figure 69 shows the curves of distribution function (9.12) for several fixed values of $\frac{a}{\sigma}$. Curve 1 corresponds to (9.13), i.e., to a purely random process.

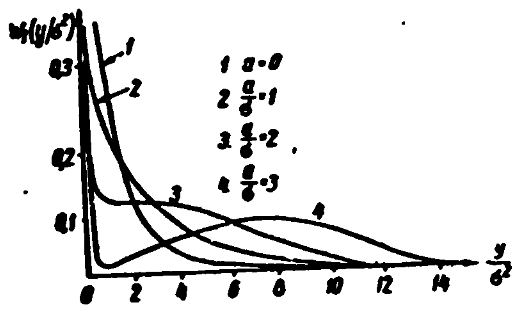


Fig. 69. Distribution function of a square of normal random process.

The expressions (9.10) and (9.12) obtained above represent the distribution functions of the square of a normal random process in the most general form, when no supplementary assumptions are made with respect to the shape of its power spectrum. For the problem under consideration in this chapter, this is equivalent to the absence of any special restrictions with respect to the shape and central frequency of the frequency characteristic of the input linear system of the standard section. If,

however, this linear system is such that its linear characteristic is symmetrical with respect to the central frequency ω_0, and its band width is $\Delta \ll \omega_0$, then the normal random process at the output of such a system will be a narrow-band one. Then its square will consist of two terms: a low-frequency one, coinciding with the square $E^2(t)$ of the envelope of a normal process, and a high-frequency one, equal to $E^2(t) \cos 2\left[\omega_0 t - \varphi(t)\right]$. The distribution function of $E^2(t)$ were determined in Sect. 4, Ch.8 [cf. (8.18) and (8.20)]. The distribution of the high-frequency component may be obtained from a consideration of the product $E^2(t) \cos 2\left[\omega_0 t - \varphi(t)\right]$. The low-frequency term of the square of the envelope of a stationary normal random process, $E^2(t)$, is distributed according to the exponential law, with the phases equiprobable. Employing (3.25) and taking into account (8.20') and (8.80'), we find the one-dimensional distribution function $W_{12}(y)$ of the high-frequency term

$$W_{12}(y)=\frac{1}{2\pi\sigma^2}\int_{|y|}^{\infty}\frac{e^{-\frac{z}{2\sigma^2}}}{z\sqrt{1-\left(\frac{y}{z}\right)^2}}\,dz=$$

$$=\frac{1}{2\pi\sigma^2}\int_{1}^{\infty}\frac{e^{-\frac{|y|}{2\sigma^2}x}}{\sqrt{x^2-1}}\,dx=\frac{1}{\pi\sigma^2}K_0\left(\frac{|y|}{\sigma}\right), \qquad (9.15)$$

where $K_0(z)$ is the Bessel function of an imaginary argument of the second kind and of a zero order (cf. G. N. Vatson (Watson), Teoriya besselevykh funktsiy (A Treatise on the Theory of Bessel Functions). Foreign Literature Pub. Hse., 1949, p. 200). A curve of the function $W_{12}(\frac{y}{\sigma})$ is shown in Figure 70.

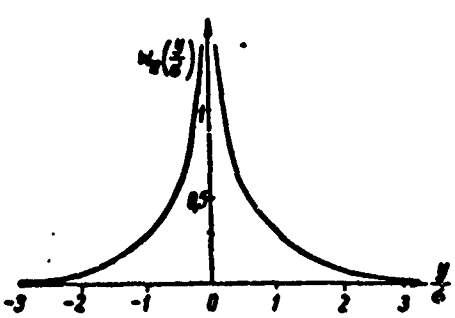


Fig. 70. Distribution function of the high-frequency term of the square of a normal random process.

3. One Result from the Theory of Integral Forms

Let us now return to the study of integral transformation (9.5); for this it

will be necessary to draw upon one result from the theory of integral forms. Omitting here the proof of this result which is referred to one of the handbooks on integral equations (cf. e.g., I. G. Petrovskiy, Lektsii po teorii integral'nykh uravneniy (Lectures on the theory of integral equations), Gostekhizdat, 1948), we shall restrict ourselves here to several geometrical analogies.

Let us consider the continuous function $y = f(x)$, in the interval (a,b). For a full determination of this function it is necessary to assign to it values at every point in the indicated interval. However, some concept of this function is provided by its values at n points $x_1, x_2, \ldots x_n$. Let us adopt the designation $y_i = f(x_i)$ ($i = 1, 2, \ldots n$). The numbers $y_1, y_2, \ldots y_n$ may be regarded as the components, in an n-dimensional space, of a vector drawn from the origin of the coordinates. Thus to the function $f(x)$ there corresponds the vector $(y_1, y_2, \ldots y_n)$. The greater is n, the more precisely is the function approximated by this vector. The length (or the "norm") of this vector is equal to $\sqrt{y_1^2 + y_2^2 + \ldots y_n^2}$. Taking the limit when $n \to \infty$, it is natural to call the "length" ("norm") of the function $f(x)$ the magnitude $\sqrt{\int_a^b f^2(x)\,dx}$. The function is called normalized, if its norm is equal to 1. The scalar product of the two vectors $(y_1^{(1)}, y_2^{(1)}, \ldots y_n^{(1)})$ and $(y_1^{(2)}, y_2^{(2)}, \ldots y_n^{(2)})$ is provided by the formula $\sum_{i=1}^{n} y_i^{(1)} y_i^{(2)}$. analogously, we shall call the scalar product of the two functions $f_1(x)$ and $f_2(x)$ the integral $\int_a^b f_1(x) f_2(x)\,dx$. Two vectors are orthogonal, if their scalar product is equal to zero. The condition for the orthogonality of the functions is written in the form of

$$\int_a^b f_1(x) f_2(x)\,dx = 0.$$

The equation for a 2-nd order surface in an n-dimensional space is put quadratically and has the form

$$\sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} y_i y_j = \text{const},$$

with $k_{ij} = k_{ji}$. The corresponding analogue in the space of the functions is the integral form

$$\int_a^b \int_a^b K(u, v) f(u) f(v)\,du\,dv = \text{const},$$

where the nucleus K(u,v) is symmetrical, i.e., $K(u,v) \equiv K(v,u)$.

It is well known that the equation for a second-order surface

$$\sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij} y_i y_j = \text{const}$$

may be reduced to the canonical form:

$$\sum_{i=1}^{m} \frac{z_i^2}{\lambda_i} = \text{const} \ (m < n),$$

if the transformation is made to such a system of coordinates, in which the principal axes of the surface serve as the coordinate axes. This reduction to the canonical form is effected by the linear transformation

$$z_i = \sum_{j=1}^{n} a_{ij} y_j,$$

the vectors $(a_{i1}, a_{i2}, \ldots a_{in})$ forming an aggregate $(i = 1, 2, \ldots n)$ of orthogonal, normalized vectors, directed along the principal semiaxes of the surface under consideration. To the true semiaxes there correspond $\lambda_i > 0$. If $m < n$, the surface degenerates into a cylindrical one. It is proved that the coefficients a_{ij} of the transformation satisfy the system of linear homogeneous equations

$$a_{ir} = \lambda_i \sum_{j=1}^{n} k_{rj} a_{ij}.$$

Completely analogously, an integral form with a symmetrical nucleus may be represented in the form of the finite or infinite summation*

$$\int_a^b\int_a^b K(u, v) f(u) f(v)\, du\, dv = \sum_{i=1} \frac{\psi_i^2}{\lambda_i}, \tag{9.16}$$

where

$$\psi_i = \int_a^b f(x) \varphi_i(x)\, dx, \tag{9.17}$$

and the functions $\varphi_i(x)$, $(i = 1, 2, \ldots)$ are an aggregate of orthogonal, normalized functions, each of which satisfies the homogeneous integral equations

$$\varphi_i(x) = \lambda_i \int_a^b K(x, y) \varphi_i(y)\, dy, \quad i = 1, 2, \ldots \tag{9.17'}$$

* When the upper limit of a summation may be finite or infinite, only its lower limit is indicated.

The solutions for $\varphi_i(x)$ are called proper orthogonal functions, and the numbers λ_i – the proper values of integral equation (9.17'). If a large number of the proper functions is finite, the nucleus of the integral equation is called a degenerate one.

Formula (9.16), and (9.17) and (9.17') which are linked to it, represent that result from the theory of integral forms which is employed for studying the distribution function of the random process at the output of a filter.

This formula also follows from the possibility of expanding the symmetrical nucleus K(u,v) into the series

$$K(u, v)=\sum_{i=1} \frac{\varphi_i(u)\varphi_i(v)}{\lambda_i}. \tag{9.18}$$

Let us note that the expansions (7.5), (8.10) and (8.25) of symmetrical two-dimensional probability densities are expansions of the type of (9.18).

4. Characteristic Function of Random Process at Output of Filter

In Section 1 it was shown that the random process at the output of a filter is represented by the integral form

$$\xi_2(t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(u, v)[S(t-u)+\xi(t-u)][S(t-v)+ \\ +\xi(t-v)]\,du\,dv,$$

the nucleus K(u,v) of which, as is not difficult to see from (9.6), is symmetrical. Then, employing (9.16), we represent the preceding expression in the form of the summation

$$\xi_2(t)=\sum_{i=1} \frac{[S_i(t)+\eta_i(t)]^2}{\lambda_i}, \tag{9.19}$$

where, in accordance with (9.17) and (9.17'),

$$S_i(t)=\int_{-\infty}^{\infty} S(t-x)\varphi_i(x)\,dx, \tag{9.20}$$

$$\eta_i(t)=\int_{-\infty}^{\infty} \xi(t-x)\varphi_i(x)\,dx, \tag{9.21}$$

and $\varphi_i(x)$ and λ_i are the proper orthogonal functions and the proper values of the homogeneous integral functions.

$$\varphi(x)=\lambda\int_{-\infty}^{\infty}K(x,y)\varphi(y)\,dy. \tag{9.22}$$

The nucleus of integral equation (9.22) is expressed in terms of the pulse tranfer functions of the rf-amplifier and filter according to formula (9.6).

Equation (9.22) may be reduced to a different equation, the nucleus of which is the product of the correlation coefficient of the noise at the output of the rf-amplifier and the pulse transfer function of the video filter. For this we substitute (9.6) into (9.22) and change the order of integration

$$\varphi(x)=\lambda\int_{-\infty}^{\infty}h_2(\tau)\,h_1(x-\tau)\int_{-\infty}^{\infty}h_1(y-\tau)\,\varphi(y)\,dy\,d\tau.$$

Having multiplied both parts of the last equality by $h_1(x - z)$ and integrating with respect to x, we obtain

$$f(z)=\lambda\int_{-\infty}^{\infty}h_2(\tau)f(\tau)\int_{-\infty}^{\infty}h_1(x-\tau)\,h_1(x-z)\,dx\,d\tau,$$

where there is designated $f(z)=\int_{-\infty}^{\infty}\varphi(x)\,h_1(x-z)\,dx.$ But according to (6.13') the correlation coefficient of white noise at the output of an rf-amplifier is equal to

$$R(\tau)=\int_{-\infty}^{\infty}h_1(u)\,h_1(u+\tau)\,du.$$

Consequently, $f(z)$ satisfies the integral equation

$$f(z)=\lambda\int_{-\infty}^{\infty}R(z-\tau)\,h_2(\tau)\,f(\tau)\,d\tau. \tag{9.22'}$$

The problem of the study of the statistical characteristics of the random process $\xi_2(t)$ is now reduced to a determination of the distribution functions of the sum of the squares of random processes $S_i(t) + \eta_i(t)$, in which $S_i(t)$ are determined, and $\eta_i(t)$ are random. Since we are restricted to the determination only of a one-dimensional distribution function, it is sufficient to carry out all further investigation in an arbitrary, but fixed moment of time t. We shall show that the random variables $\eta_k(t)$ and $\eta_j(t)$ when $k \neq j$ are uncorrelated. This makes it possible for us merely to define the joint determination of $\eta_1(t)$, $\eta_2(t)$, ... $\eta_n(t)$ as the product of their one-dimensional distribution functions.

Let us then examine the mean value of the product

$$m_1\{\eta_k(t)\eta_j(t)\}=m_1\{\int_{-\infty}^{\infty}\xi(t-x)\varphi_k(x)dx\int_{-\infty}^{\infty}\xi(t-y)\varphi_j(y)dy\}=$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_k(x)\varphi_j(y)m_1\{\xi(t-x)\xi(t-y)\}dxdy.$$

Bearing in mind that the correlation function of white noise is equal [cf. (5.84)] to

$$m_1\{\xi(t)\xi(t+\tau)\}=\sigma^2\delta(\tau)$$

and taking into account the filtering property of the delta-function (cf. Appendix IV), we find

$$m_1\{\eta_k(t)\eta_j(t)\}=\sigma^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_k(x)\varphi_j(y)\delta(x-y)dxdy=$$

$$=\sigma^2\int_{-\infty}^{\infty}\varphi_k(x)\varphi_j(x)dx.$$

But the functions $\varphi_i(x)$ are mutually orthogonal, therefore

$$m_1\{\eta_k(t)\eta_j(t)\}=\begin{cases}\sigma^2 & k=j\\ 0 & k\neq j.\end{cases} \quad (9.23)$$

Q.E.D.

Since it is a condition of the problem that the white noise $\xi(t)$ at the output of the rf-amplifier has a normal distribution, the distributions of $\eta_i(t)$, represented by integrals of $\xi(t)$, will also be normal. Then in virtue of (9.23) their joint distribution will be equal to the product

$$W(y_1, y_2, \ldots y_n, \ldots)=\prod_{i=1}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{y_i^2}{2\sigma^2}}. \quad (9.24)$$

Having the product (9.24), it is now not difficult to obtain the expression for the desired one-dimensional distribution function of the random process $\xi_2(t)$ at th output of the filter. For this there should first be determined the characteristic function $\Theta_1(v,t)$ of summation (9.19). Employing formulas (3.73) and (3.76), we fin

$$\Theta_1(v,t)=\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}e^{iv\sum_j\frac{[S_j(t)+y_j]^2}{\lambda_j}}W(y_1, y_2, \ldots y_n, \ldots)\times \quad (9.25)$$

$$\times dy_1 dy_2 \ldots dy_n \ldots = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_j \Big\{ e^{\frac{iv}{\lambda_j}(S_j^2 + 2S_j y_j + y_j^2)} \times$$

$$\times \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y_j^2}{2\sigma^2}} dy_j \Big\} = \qquad \text{(9.25) (cont'd)}$$

$$= \prod_j \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{ivs_j^2}{\lambda_j}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(1 - 2i\sigma^2 \frac{v}{\lambda_j}\right) y_j^2 + \frac{2iS_j v}{\lambda_j} y_j} dy_j.$$

Completing the square of the exponent under the integral and integrating (cf. Sect. 8, Chapter III), we obtain after simple algebraic transformations

$$\Theta_1(v, t) = \prod_j \frac{1}{\sqrt{1 - \frac{2iv\sigma^2}{\lambda_j}}} e^{\frac{s_j^2}{2\sigma^2} \cdot \frac{2iv\sigma^2}{\lambda_j - 2iv\sigma^2}}. \qquad (9.26)$$

For the case of a narrow-band process, the expression for the envelope $E_2(t)$ at the output of the filter is, in accordance with (9.7), obtained from (9.19) by the replacement of $[S_i(t) + \eta_i(t)]^2$ by the sum of squares

$$E_2(t) = \sum_i \frac{1}{\lambda_i} \{[S_{Ai}(t) + \eta_{Ai}(t)]^2 + [S_{Ci}(t) + \eta_{Ci}(t)]^2\}, \qquad (9.27)$$

where

$$S_{Ai}(t) = \int_{-\infty}^{\infty} S_A(t - x) \varphi_i(x), dx, \; S_{Ci} = \int_{-\infty}^{\infty} S_C(t - x) \varphi_i(x) dx, \qquad (9.28)$$

$$\eta_{Ai} = \int_{-\infty}^{\infty} \xi_A(t - x) \varphi_i(x) dx, \;\; \eta_{Ci} = \int_{-\infty}^{\infty} \xi_C(t - x) \varphi_i(x) dx. \qquad (9.29)$$

Then the one-dimensional characteristic function of this envelope is obtained analogously to (9.25)

$$\Theta_1(v, t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\Big\{ iv \sum_j \frac{1}{\lambda_j} [(S_{Aj} + x_j)^2 + (S_{Cj} + y_j)^2] \Big\} \times$$

$$\times \prod_j \Big\{ \frac{1}{2\pi\sigma^2} e^{-\frac{x_j^2 + y_j^2}{2\sigma^2}} dx_j dy_j \Big\} = \qquad (9.30)$$

$$= \Big\{ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\Big\{ iv \sum_j \frac{1}{\lambda_j} (S_{Aj} + x_j)^2 \Big\} \prod_j \Big[\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x_j^2}{2\sigma^2}} dx_j \Big] \Big\} \times$$

$$\times \Big\{ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\Big\{ iv \sum_j \frac{1}{\lambda_j} (S_{Cj} + y_j)^2 \Big\} \prod_j \Big[\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y_j^2}{2\sigma^2}} dy_j \Big] \Big\}.$$

Each of the multiple integrals in the curved brackets coincides with (9.25), if only S_j is replaced by S_{Aj} or respectively by S_{Cj}. Therefore, taking into account (9.26), we find

$$\Theta_1(v, t)=\prod_j \frac{1}{1-\frac{2iv\sigma^2}{\lambda_j}} e^{\frac{S_{Aj}^2+S_{Cj}^2}{2\sigma^2}\cdot\frac{2iv\sigma^2}{\lambda_j-2iv\sigma^2}}. \tag{9.31}$$

The desired one-dimensional distribution function of the random process (or envelope) at the output of a filter is obtained from (9.26) or respectively from (9.31) by an inverse Fourier transformation.

In principle, the problem before us appears to be solved. However, in formulas (9.26) and (9.31) there figure the characteristic numbers λ_j, for the determination of which it is still necessary to solve the integral equation (9.22) or (9.22'). Only in one special case does the solution of this integral equation turn out to be extremely simple. This is the case of an output filter whose frequency characteristic is uniform at all frequencies. In this case $h_2(\tau) = \delta(\tau)$, and from (9.6), taking into account the filtering property of the delta-function, we find

$$K(u, v)=h_1(u)\,h_1(v). \tag{9.32}$$

Comparing (9.32) with (9.18), we become convinced that the nucleus is degenerate, since there corresponds to it only one proper value of λ and one proper function $\varphi(u) = \sqrt{\lambda}\; h_1(u)$ with λ being determined from the condition that $\varphi(u)$ is normalized, i.e., that

$$\lambda\int_{-\infty}^{\infty} h_1^2(u)\,du=\frac{\lambda}{2\pi}\int_{-\infty}^{\infty} C_1^2(\omega)\,d\omega=1.$$

The relationship $\frac{\sigma^2}{\lambda}=\frac{\sigma^2}{2\pi}\int_{-\infty}^{\infty} C_1^2(\omega)\,d\omega=\sigma_1^2$ thus constitutes the dispersion of noise at the output of an rf-amplifier.

From (9.26) we find in this case

$$\Theta_1(v, t)=\frac{1}{\sqrt{1-2iv\sigma_1^2}}\, e^{\frac{S_1^2(t)}{2\sigma^2}\,\frac{2iv\sigma_1^2}{1-2iv\sigma_1^2}}, \tag{9.33}$$

where $S_1^2(t)$ is a signal which has passed through the rf-amplifier.

Since the output filter has an unrestricted band, formula (9.33) yields the expression for the characteristic function of the square of a normal random process with a dispersion of σ_1^2. An inverse Fourier transformation of it coincides with (9.12) (with, of course, the appropriate replacement of σ by σ_1, and of s_1 by a).

From (9.31) we have in this case

$$\Theta_1(v, t) = \frac{1}{1 - 2iv\sigma_1^2} e^{+\frac{a^2(t)}{2\sigma^2} \cdot \frac{2iv\sigma_1^2}{1-2iv\sigma_1^2}}, \tag{9.34}$$

where α (t) is the envelope of a signal at the output of the rf-amplifier. Formula (9.34) yields the expression for the characteristic function of the square of the envelope of a normal random process. Its inverse Fourier transformation coincides with (3.20).

Let us note that the method set forth above may be generalized and employed for calculation of the multi-dimensional characteristic functions of a random process at the output of a standard section of the type under consideration [1].

5. An Approximate Method of Determining the Distribution Function

With the exception of one special case, cited at the end of the preceding section, the solution of integral equation (9.22) is a sufficiently laborious process. Since, besides, in the majority of cases in practice this solution is obtained by approximate methods, it is worth our while to consider the approximate methods of directly determining the distribution function of a process at the output of a filter, avoiding the stage of solving integral equation (9.22). One such method, consisting in the computation of a number of distribution moments, was cited in §8, Chapter VI. This method may be applied in detail to the problem considered in the present chapter, if account is taken of (9.26) or (9.31).

Let us regard as numerical characteristics of the random processes at the output of a filter not the moments, but the cumulants (cf. p. 109)of one-dimensional distribution.

By definition a n-th-order cumulant is equal to

$$k_n = i^n \left[\frac{d^n}{dv^n} \ln \Theta_1(v)\right]_{v=0}.$$

From (9.26) we find

$$\ln \Theta_1(v, t) = -\frac{1}{2}\sum_j \ln\left(1 - 2iv\,\frac{\sigma^2}{\lambda_j}\right) + \sum_j \frac{S_j^2(t)}{2\sigma^2}\cdot\frac{2iv\sigma^2}{\lambda_j - 2iv\sigma^2},$$

wherefrom by consecutive differentiation we find the n-th-order cumulant of random process $\xi_2(t)$:

$$k_n(t) = (2\sigma^2)^n \frac{(n-1)!}{2} \sum_j \frac{1}{\lambda_j^n} + (2\sigma^2)^{n-1} n! \sum_j \frac{S_j^2(t)}{\lambda_j^n}. \tag{9.35}$$

In exactly the same manner, from (9.31) there can be found the cumulant of the envelope $E_2(t)$ of the process at the output of a filter

$$k_{n0}(t) = (2\sigma^2)^n (n-1)! \sum_j \frac{1}{\lambda_j^n} + (2\sigma^2)^{n-1} n! \sum_j \frac{S_{Aj}^2(t) + S_{Cj}^2(t)}{\lambda_j^n}. \tag{9.36}$$

The series entering into (9.35) and (9.36) may be expressed through iteration of the nucleus $K^{(n)}(u,v)$, and that itself excludes the proper numbers λ_j. The iterated nucleus $K^{(n)}(u,v)$, is obtained from the basic nucleus $K(u,v)$ by means of (n-1) integrations

$$K^{(n)}(u, v) = \underbrace{\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty}}_{n-1 \text{ раз}} K(u, x_1) K(x_1, x_2) \dots K(x_{n-1}, v)\, dx_1 dx_2 \dots dx_{n-1}. \tag{9.37}$$

Substituting under the integral sign of (9.37) in place of $K(x_K, x_{K+1})$ its expansion (9.18), and bearing in mind that the aggregate of functions $\varphi_i(x_K)$ is orthogonal and normalized, we obtain

$$\int_{-\infty}^{\infty} K^{(n)}(u, u)\, du = \sum_j \frac{1}{\lambda_j^n}. \tag{9.38}$$

Analogously

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(t-v) S(t-u) K^{(n)}(u, v)\, du dv = \sum_j \frac{S_j^2(t)}{\lambda_j^n}. \tag{9.39}$$

Substituting (9.38) and (9.39) into (9.35), we find

$$k_n(t) = (2\sigma^2)^n \frac{n!}{2}\left\{\frac{1}{n}\int_{-\infty}^{\infty} K^{(n)}(u, u)\,du + \right.$$

$$\left. + \frac{1}{\sigma^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(t-u)K^{(n)}(u, v)S(t-v)\,du\,dv\right\}. \tag{9.40}$$

Thus, employing (9.40), it is possible to determine arbitrary-order cumulants of the one-dimensional distribution function of a random process at the output of a filter, without solving the integral equation. It is not difficult to write an expression, analogous to (9.40), for the cumulant of the envelope as well. If the signal is absent, then the double integral in (9.40) disappears.

After some number of the cumulants of a random process has been found, the question arises as to the means of their employment in the approximate determination of a one-dimensional distribution function. The desired distribution function is for this purpose usually represented in the form of a resolution into assigned orthogonal functions; the coefficients of this resolution are expressed in terms of the distribution cumulants. As the system of orthogonal functions into which the resolution takes place, there is usually taken the function $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ and its derivatives (Grammat-Charlier series, Edgeworth series, cf. G. Kramer [i.e., Harold Cramer]. Matematicheskiye metody statistiki [Mathematical Methods of Statistics], Moskva, For. Lit. Pub. Hse., 1948). Sometimes it is more convenient to employ a resolution of the distribution function into a series on the basis of the Laguerre functions [2].

Let us cite here an expansion into an Edgeworth series. Let $w_1(x)$ be the desired distribution function of the random process at the output of a filter, and k_1, k_2, k_3 be the first three cumulants of this distribution. Then the first four terms of the expansion of $w_1(x)$ into an Edgeworth series have the form of

$$w_1(x) = \frac{1}{\sqrt{k_2}}\left[\varphi\left(\frac{x-k_1}{\sqrt{k_2}}\right) - \frac{1}{3!}\frac{k_3}{k_2^{3/2}}\varphi^{(3)}\left(\frac{x-k_1}{\sqrt{k_2}}\right) + \right.$$

$$\left. + \frac{1}{4!}\frac{k_4}{k_2^2}\varphi^{(4)}\left(\frac{x-k_1}{\sqrt{k_2}}\right) + \frac{10}{6!}\frac{k_3^2}{k_2^3}\varphi^{(6)}\left(\frac{x-k_1}{\sqrt{k_2}}\right) + \dots\right]. \tag{9.41}$$

where $\varphi^{(3)}$, $\varphi^{(4)}$, $\varphi^{(6)}$ are derivatives of $\varphi(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, and the magnitudes $\kappa_3/\kappa_2^{3/2}$ and κ_4/κ_2^2 coincide with the coefficients of asymmetry and excess of distribution $w_1(x)$ [cf. (3.68)].

Let us note that the Edgeworth series has already been employed earlier in §2, Chapter IV in an evaluation of the rapidity of convergence of the distribution function of a sum of independent random variables with a normal one. A comparison of (9.41) with (4.19) establishes the complete identity of the two equalities.

Figure 71 shows graphs of the derivatives $\varphi^{(3)}(x)$, $\varphi^{(4)}(x)$, and $\varphi^{(6)}(x)$ with such coefficients as they have when they enter into expansion (9.41). The curves of $\varphi^{(4)}(x)$ and $\varphi^{(6)}(x)$ are symmetrical with respect to x = 0, and the third derivative $\varphi^{(3)}(x)$ introduces an asymmetrical element into the expression of the distribution function.

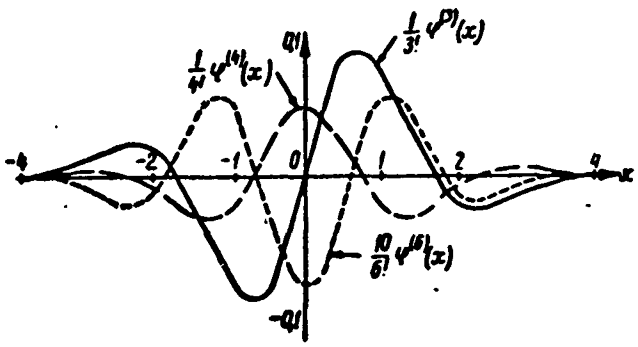


Fig. 71. Derivatives of the function $\varphi(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$

6. Example of Calculation of Distribution Function of Process at Filter Output

As an illustration of the method set forth above, we cite the example of calculating the distribution function of a random process at the output of a standard section consisting of an rf-amplifier, a square-law detector and a video filter [2].

Let the frequency characteristics of the rf-amplifier and the filter be symmetrical and describable by the gaussian curves*

$$C_1(\omega)=e^{-\frac{(\omega-\omega_0)^2}{2\gamma_1^2}}+e^{-\frac{(\omega+\omega_0)^2}{2\gamma_1^2}},$$
$$C_2(\omega)=e^{-\frac{\omega^2}{2\gamma_2^2}}.$$

* Cf. note on p. 229 concerning the physical feasibility of linear systems with assigned frequency characteristics.

The corresponding pulse transfer functions have the form of

$$h_1(\tau)=\frac{1}{\sqrt{2\pi}}\beta_1 e^{-\frac{\beta_1^2\tau^2}{2}}\cos\omega_0\tau, \tag{9.42}$$

$$h_2(\tau)=\frac{1}{\sqrt{2\pi}}\beta_2 e^{-\frac{\beta_2^2\tau^2}{2}}. \tag{9.42'}$$

The parameters β_1 and β_2 are simply expressed in terms of band Δ_1 of the rf-amplifier and of band Δ_2 of the filter (cf. p. 229)

$$\Delta_1=\sqrt{\pi}\beta_1,\ \Delta_2=\frac{1}{2}\sqrt{\pi}\beta_2. \tag{9.43}$$

We introduce designations for the ratio of these parameters

$$\nu=\frac{\beta_2}{\beta_1}=\frac{2\Delta_2}{\Delta_1}. \tag{9.44}$$

Substituting (9.42) and (9.42') into (9.6) and performing the integration, we find

$$K(u, v)=\frac{4\pi\beta_1\beta_2\cos\omega_0(u-v)}{\sqrt{\nu^2+2}}e^{-\pi^2\beta_1^2\left[(u+v)^2+\frac{\nu^2}{\nu^2+2}(u-v)^2\right]}. \tag{9.45}$$

The iterated second-order nucleus is, in accordance with (9.37), equal to

$$K^{(2)}(u, v)=\int_{-\infty}^{\infty}K(u, x)K(x, v)\,dx=\frac{4\pi^{3/2}\beta_1\beta_2^2\cos\omega_0(u-v)}{\sqrt{(\nu^2+2)(\nu^2+1)}}\times$$
$$\times e^{-\pi^2\beta_1^2\left[\frac{\nu^2+1}{\nu^2+2}(u-v)^2+\frac{\nu^2}{\nu^2+1}(u+v)^2\right]}. \tag{9.46}$$

Analogously the iterated third-order nucleus is

$$K^{(3)}(u, v)=\int_{-\infty}^{\infty}K^{(2)}(u, y)K^{(2)}(y, v)\,dy=\frac{8\pi^2\beta_1\beta_2^3\cos\omega_0(u-v)}{\sqrt{(\nu^2+2)(2\nu^2+1)(2\nu^2+3)}}\times$$
$$\times e^{-\pi^2\beta_1^2\left[\frac{2\nu^2+1}{2\nu^2+3}(u-v)^2+\frac{(2\nu^2+3)\nu^2}{(2\nu^2+1)(\nu^2+2)}(u+v)^2\right]}. \tag{9.47}$$

Precisely expressed, the nucleus K(u,v) and its iterations contain second items, which are small to the point of disappearance if $\omega_0 \gg \Delta_1$.

Employing (9.45) - (9.47), we find

$$\int_{-\infty}^{\infty}K(u, u)\,du=2\beta_1\sqrt{\pi}=\Delta_1, \tag{9.48}$$

$$\int_{-\infty}^{\infty}K^{(2)}(u, u)\,du=\frac{\nu\Delta_1^2}{2\sqrt{\nu^2+2}}, \tag{9.49}$$

$$\int_{-\infty}^{\infty}K^{(3)}(u, u)\,du=\frac{\nu^2\Delta_1^3}{2(2\nu^2+3)}. \tag{9.50}$$

Let us assume that the determined part of the process at the input of a standard section constitutes a harmonic signal with a constant amplitude of $S(t) = A\cos\omega_0 t$ (the frequency ω_0 coincides with the resonance frequency of the rf-amplifier). Then, taking into account (9.45) - (9.47), we find that the double integrals in (9.40) are, for the example under consideration, equal to

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(t-u)K(u,v)S(t-v)\,du\,dv = \frac{A^2}{2}, \tag{9.51}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(t-u)K^{(2)}(u,v)S(t-v)\,du\,dv = \frac{\nu\Delta_1}{\sqrt{\nu^2+1}}\cdot\frac{A^2}{4}, \tag{9.52}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(t-u)K^{(3)}(u,v)S(t-v)\,du\,dv = \frac{\nu^2\Delta_1^2}{\sqrt{(2\nu^2+1)(2\nu^2+3)}}\cdot\frac{A^2}{4}. \tag{9.53}$$

Substituting (9.48) - (9.53) into (9.40), we obtain cumulants of the first three orders for the random process at the output of the video filter.

The first-order cumulant (or the mean value) is equal to

$$k_1 = \sigma^2\Delta_1 + \frac{A^2}{2} = \sigma^2\Delta_1\left(1 + \frac{A^2}{2\sigma^2\Delta_1}\right). \tag{9.54}$$

The second-order cumulant (or the dispersion) is equal to

$$k_2 = 2\sigma^4\frac{\nu\Delta_1^2}{2\sqrt{\nu^2+2}} + 4\sigma^2\frac{\nu\Delta_1}{\sqrt{\nu^2+1}}\cdot\frac{A^2}{4} = \frac{\nu\sigma^4\Delta_1^2}{\sqrt{\nu^2+2}}\left(1 + \frac{A^2}{\sigma^2\Delta_1}\sqrt{\frac{\nu^2+2}{\nu^2+1}}\right). \tag{9.55}$$

The third-order cumulant (or the third-order central moment) is equal

$$k_3 = 8\sigma^6\frac{\nu^2\Delta_1^3}{2(2\nu^2+3)} + 24\sigma^4\frac{\nu^2\Delta_1^2}{\sqrt{(2\nu^2+1)(2\nu^2+3)}}\frac{A^2}{4} = \frac{4\nu^2\sigma^6\Delta_1^3}{2\nu^2+3}\left(1 + 3\frac{A^2}{2\sigma^2\Delta_1}\sqrt{\frac{2\nu^2+3}{2\nu^2+1}}\right). \tag{9.56}$$

The coefficient of asymmetry of the one-dimensional distribution function of a random process at the output of the filter is equal to

$$k = \frac{k_3}{k_2^{3/2}} = 4\sqrt{\nu}\,\frac{1+3a^2\sqrt{\frac{2\nu^2+3}{2\nu^2+1}}}{2\nu^2+3}\left(\frac{\sqrt{\nu^2+2}}{1+2a^2\sqrt{\frac{\nu^2+2}{\nu^2+1}}}\right)^{3/2}, \tag{9.57}$$

where $a = \frac{A}{\sigma\sqrt{2\Delta_1}}$ is the ratio of effective value of the signal to the effective value of the noise at the output of the rf-amplifier.

It could be possible, in a completely analogous manner, to continue to compute cumulants of a higher order. We cite merely the final result:

$$k_n = \frac{(2\sigma^2\Delta_1)^n (n-1)!}{(\nu+\sqrt{\nu^2+2})^n - (\sqrt{\nu^2+2}-\nu)^n} \times$$
$$\times\left(1 + n\frac{A^2}{2\sigma^2\Delta_1}\sqrt{\frac{(\sqrt{\nu^2+2}+\nu)^n - (\sqrt{\nu^2+2}-\nu)^n}{(\sqrt{\nu^2+2}+\nu)^n + (\sqrt{\nu^2+2}-\nu)^n}}\sqrt{\frac{\nu^2+2}{\nu^2}}\right). \tag{9.58}$$

Having the magnitudes of the cumulants, and expanding into a series in terms of orthogonal functions [e.g., series (9.41)], it is possible with an assigned degree of precision to plot the distribution function of a random process at the output of the video filter.

Figure 72 shows the curves, obtained by the indicated method, of the one-dimensional distribution function of the process at the output of a standard section, for the case when only "white" noise is applied to its input. To each curve there corresponds a constant ratio v of the width of the filter band Δ_2 to half the width of the rf-amplifier band $\frac{\Delta_1}{2}$.

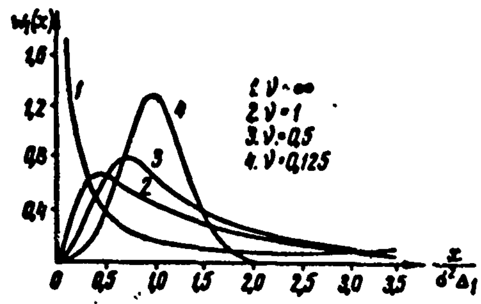


Fig. 72. Distribution function of process at output of standard section (s = 0)

In Figures 73 and 74 these same curves are plotted for cases where there is acting a sinusoidal signal as well, the ratio $\frac{A^2}{2\sigma^2\Delta_1} = s^2$ being equal, respectively, to one and to two. When $v \to \infty$ (curves 1 in Figs. 72, 73, 74), i.e., as the filter band width increases, the curves of the distribution functions approach the respective curves shown in Figure 69.

It can be seen from the curves shown that, as the filter band grows narrower ($v \to 0$), the distribution functions approach the normal. This normalization of a random process at the output of a narrow-band linear system is, as has been noted in

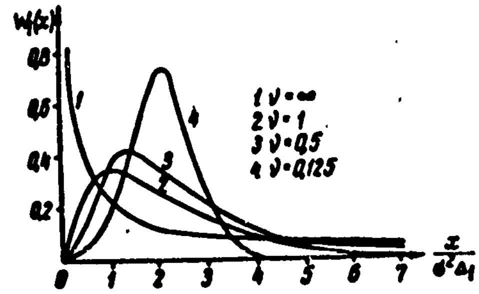


Fig. 73. Distribution function of process at output of standard section (s = 1).

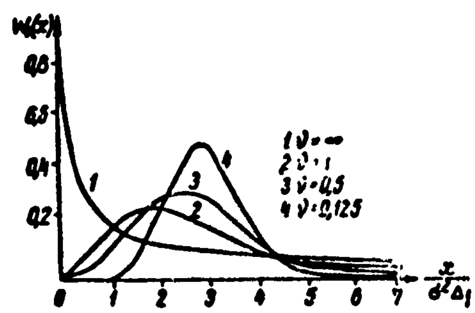


Fig. 74. Distribution function of process at output of standard section (s = 2).

Sect. 8, Ch. VI, a consequence of the central limit theorem. The tendency toward normalization increases with an increase in the signal/noise ratio s at the output of the rf-amplifier.

In the first approximation the degree of normalization of a random process at the output of a filter may be evaluated by means of the coefficient of asymmetry k, the dependence of which on v and s is given by formula (9.57) and by the graphs corresponding to it in Figure 75.

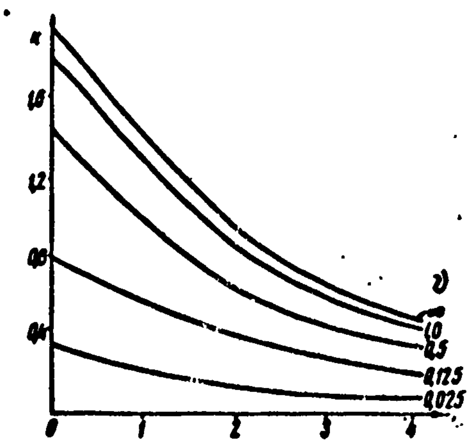


Fig. 75. Coefficient of asymmetry of process at output of standard section, in relation to the ratio $s = \frac{A}{\sigma\sqrt{2\Delta_1}}$ with fixed value of $v = \frac{2\Delta_2}{\Delta_1}$.

With small ν and a constant s, the coefficient of asymmetry is, as follows from (9.57), proportional to $\sqrt{\nu} = \sqrt{\frac{2A_2}{\Delta_1}}$. This coincides with the result obtained in [6], by the method set forth in Sect. 8, Ch. VI, for a standard link which consists of an rf-amplifier, a linear detector and a narrow-band filter, the frequency characteristics of the rf-amplifier and the filter being rectangular.

Literature.

1. M. A. Mayer, D. Middleton. On the distributions of signals and noise after rectification and filtering. Journ. Appl. Phys. 25, No. 8, 1954.

2. R. C. Emerson. First probability for receivers with square law detectors. Journ. Appl. Phys., 24, No. 9, 1953.

3. G. R. Arthur. The statistical properties of the output of a frequency sensitive device. Journ. Appl. Phys., 25, No. 9, 1954.

4. P. I. Kuznetsov, R. L. Strotanovich, V. I. Tikhonov. The Passage of Certain Random Functions through Linear Systems. "Avtomatika i Telemekhanika", 1953, V.XIV, No. 2.

5. M. Kac, Sigert A. J. F. On the theory of noise in radio receivers with square law detectors. Journ. Appl. Phys., 18, No. 4, 1947.

6. B. R. Levin, S. A. Smirnov. On the Question of the Normalization of a Random Process in Passage through a Narrow-band System. Vestnik NII, No. 2 (47), 1954.

7. D. G. Lampard. The probability for the filtered output of a multiplier, IRE Transaction, IT—2, No. 1, March 1956.

CHAPTER X

POWER SPECTRA OF SIGNALS, MODULATED BY RANDOM PROCESSES

1. The Pulse Random Process

In pulse engineering, which has undergone considerable development in recent years, many problems lead to an investigation of the sequence spectra of identical pulses. The basic parameters, characterizing the geometric shape or position of these pulses (amplitude, duration, instant of origin of leading edge, etc.) can change in accordance with a given law or can be random functions of time. The latter takes place when the pulses are distorted by random interference, or when the modulation of the pulse sequence may be regarded as a random process. A sequence of pulses whose parameters are random variables we shall call a pulse random process.

If the pulse shape is given and one of its geometric parameters is random, then to the pulse sequence there corresponds a sequence of random variables, namely: to the beginning of each cadence interval there may be assigned a random value of the pulse parameter. Such a random sequence represents a special case of a random process with discrete time, the theory of which is developed in parallel with the theory of random process with continuous time.

A pulse random process is generally non-stationary. Thus if the moments of pulse emergence are periodic, and the parameters characterizing the geometric shape are random, then it is obvious that the two values of a pulse random process, at the moment of passage of a pulse and in the interval between pulses, are independent. The values of a random process may become statistically related, if two moments in time are examined with reference to the passage of an aribtrary pair of pulses. Finally, the value of the examined random function is uniquely determined for the interval between the pulses.

Thus the correlation coefficient of a pulse random process can, with a given magnitude τ of the difference of two instants in time, take any value from zero to

one. The mean value of a pulse random process also depends on time. In the intervals between pulses it is always equal to zero, whereas for time instants corresponding to the passage of pulses, the magnitude of the mean value may differ from zero and may be different for various pulses.

A pulse random process is determined by an infinite number of realizations, each of which constitutes a sequence of pulses. Let us segregate one (for instance the k-th) of these sequences and examine 2N + 1 pulses, located on both sides of the zero pulse linked with the origin of the time reading.

Let us designate by $Z_{kN}(\omega)$ the spectrum density (Fourier transformation) for the function describing this sequence, and let the distance between its extreme pulses be (2N + 1)T. Since the pulse process is nonstationary, the mean power of the pulse sequence [cf. (5.42)] $G_k(\omega) = \lim_{N\to\infty} \frac{2}{(2N+1)T} |Z_{kN}(\omega)|^2$ will depend on k, i.e., on that one of the realizations of the pulse random process for which this power is computed.

In order to determine the power spectrum $F(\omega)$ of a pulse random process, it is necessary to perform a supplementary averaging of $G_k(\omega)$ for the multiplicity of realizations. Thus the power spectrum of a pulse process is determined from the relationship

$$F(\omega) = m_1 \left\{G_k(\omega)\right\} = m_1 \left\{\lim_{N\to\infty} \frac{2}{(2N+1)T} |Z_{kN}(\omega)|^2\right\}. \tag{10.1}$$

This same power spectrum may be obtained by a Fourier transformation of the correlation function, averaged over time, of a pulse random process. However in the subsequent presentation, formula (10.1) will be used to calculate the power spectrum of a random process, and the correlation function of this process is found by means of a Fourier transformation of the power spectrum $F(\omega)$.

Let us examine some realization $\xi^{(k)}(t)$ of a pulse random process.

To each pulse there may be assigned a numeral - a number (positive or negative) of the natural series. Let a pulse, belonging to this realization and emerging at the moment in time $t_n^{(k)}$, be described by the function $\xi_n^{(k)}(t - t_n^{(k)})$. This

function must satisfy the condition $\xi_n^{(k)}(t) \equiv 0$ when $t < 0$. The sequence $2N+1$ of the pulses of the examined realization may be analytically expressed by the summation

$$\sum_{n=-N}^{N} \xi_n^{(k)}(t - t_n^{(k)}).$$

Let us assume $t_n^{(k)} = nT + \nu_n^{(k)}$, where T is a positive constant and let $F_n^{(k)}(\omega)$ be the Fourier transformation of $\xi_n^{(k)}(t)$. We designate

$$V_n^{(k)} = F_n^{(k)}(\omega) e^{i\omega\nu_n^{(k)}}. \tag{10.2}$$

Then the Fourier transformation of $\xi_n^{(k)}(t - t_n^{(k)})$ will be equal to

$$F_n^{(k)}(\omega) e^{i\omega t_n^{(k)}} = V_n^{(k)} e^{in\omega T}. \tag{10.2'}$$

It is assumed that the pulses do not overlap, i.e., that in each cadence interval there emerges one pulse, and only one. This condition signifies that the possible values of the random variable ν_n and of the random duration of the n-th pulse do not exceed T/2 in absolute value.

Let us now write the spectral density (Fourier transformation) of the sequence of 2N + 1 pulses. Considering (10.2'), we find

$$Z_{kN}(\omega) = \sum_{n=-N}^{N} V_n^{(k)} e^{in\omega T}. \tag{10.3}$$

To determine the power spectrum of the pulse random process we substitute (10.3) into the general formula (10.1)

$$F(\omega) = m_1 \left\{ \lim_{N\to\infty} \frac{2}{(2N+1)T} \left| \sum_{n=-N}^{N} V_n^{(k)} e^{in\omega T} \right|^2 \right\}. \tag{10.4}$$

Changing in (10.4) the order of transition to the limit and of the averaging for the multiplicity, we obtain

$$F(\omega) = \frac{2}{T} \lim_{N\to\infty} \frac{1}{2N+1} m_1 \left\{ \left| \sum_{n=-N}^{N} V_n^{(k)} e^{in\omega T} \right|^2 \right\}. \tag{10.5}$$

It can be seen from (10.5), that to determine $F(\omega)$ it is necessary first to find the average for the multiplicity (i.e., for the index k) of $|Z_{KN}(\omega)|^2$. Since

$$|Z_{kN}(\omega)|^2 = Z_{kN}(\omega) \cdot \overline{Z_{kN}(\omega)},$$

therefore

$$|Z_{kN}(\omega)|^2 = \sum_{n=-N}^{N} \sum_{j=-N}^{N} V_n^{(k)} \bar{V}_j^{(k)} e^{i\omega(n-j)T}. \tag{10.6}$$

Here the line over V indicates a conjugate-complex quantity. Segregating in summation (10.6) the terms corresponding to n = j, we obtain

$$m_1\{|Z_{kN}(\omega)|^2\} = m_1\left\{\sum_{n=-N}^{N} |V_n^{(k)}|^2 + \sum_{\substack{n=-N \\ n \neq j}}^{N} \sum_{j=-N}^{N} V_n^{(k)} \bar{V}_j^{(k)} e^{i(n-j)\omega T}\right\},$$

and since the average of a sum is equal to the sum of the average items, therefore

$$m_1\{|Z_{kN}(\omega)|^2\} = \sum_{n=-N}^{N} m_1\{|V_n^{(k)}|^2\} + \\ + \sum_{\substack{n=-N \\ n \neq j}}^{N} \sum_{j=-N}^{N} m_1\left\{V_n^{(k)} \bar{V}_j^{(k)} e^{i(n-j)\omega T}\right\}. \tag{10.7}$$

From this point on the investigation is restricted only to such pulse processes, in which the statistical characteristics of the pulses do not depend on the numeral thereof, and the statistical characteristics of a combination of pulses depend on the relative position of the pulses and do not depend on which of them is selected as the zero one. With these restrictions the quantity

$$K(\omega) = m_1\left\{|V_n^{(k)}|^2\right\} \tag{10.8}$$

does not depend on the pulse numeral n, and the quantity

$$H_{n-j}(\omega) = m_1\left\{V_n^{(k)} \bar{V}_j^{(k)}\right\} \tag{10.9}$$

depends only on the difference n - j of the numerals of two pulses.

Then

$$\sum_{n=-N}^{N} m_1\left\{|V_n^{(k)}|^2\right\} = (2N+1)K(\omega), \tag{10.10}$$

and the double summation may, after the simplest of transformations, be represented in the form of

$$\sum_{\substack{n=-N \\ n \neq j}}^{N} \sum_{j=-N}^{N} m_1\left\{V_n^{k} \bar{V}_j^{(k)}\right\} e^{i(n-j)\omega T} = 2\sum_{p=1}^{2N} (2N+1-p) H_p(\omega) \cos p\omega T. \tag{10.11}$$

Substituting (10.10) and (10.11) into (10.7) and taking into account (10.5), we find

$$F(\omega)=\frac{2}{T}\left\{K(\omega)+\lim_{N\to\infty}2\sum_{p=1}^{2N}\left(1-\frac{p}{2N+1}\right)H_p(\omega)\cos p\omega T\right\}. \tag{10.12}$$

If the pulses are mutually independent, then

$$H_{n-l}(\omega)=m_1\left\{V_n^{(k)}\right\}m_1\left\{\bar{V}_l^{(k)}\right\}=|H(\omega)|^2, \tag{10.13}$$

where

$$H(\omega)=m_1\left\{V_n^{(k)}\right\}. \tag{10.14}$$

We designate

$$\psi(\omega)=2\lim_{N\to\infty}\sum_{p=1}^{2N}\left(1-\frac{p}{2N+1}\right)[H_p(\omega)-|H(\omega)|^2]\cos p\omega T. \tag{10.15}$$

Then formula (10.12) may be rewritten thus:

$$F(\omega)=\frac{2}{T}\left\{K(\omega)-|H(\omega)|^2+\psi(\omega)+\right.$$
$$\left.+|H(\omega)|^2\lim_{N\to\infty}\left[1+2\sum_{p=1}^{2N}\left(1-\frac{p}{2N+1}\right)\cos p\omega T\right]\right\}. \tag{10.16}$$

Noting that

$$\Lambda_N(\omega)=1+2\sum_{p=1}^{2N}\left(1-\frac{p}{2N+1}\right)\cos p\omega T=$$
$$=\frac{1}{2N+1}\sum_{n=-N}^{N}\sum_{l=-N}^{N}e^{i(n-l)\omega T}=\frac{1}{2N+1}\left\{\frac{\sin\left[(2N+1)\frac{\omega T}{2}\right]}{\sin\frac{\omega T}{2}}\right\}^2$$

and that

$$\lim_{N\to\infty}\frac{1}{2N+1}\left\{\frac{\sin\left[(2N+1)\frac{\omega T}{2}\right]}{\sin\frac{\omega T}{2}}\right\}^2=\begin{cases}0 & \omega T\neq 2\pi r,\\ \infty & \omega T=2\pi r,\end{cases}$$
$$r=0,\pm1,\pm2,\ldots$$

we find

$$\lim_{N\to\infty}\Lambda_N(\omega)=c\delta\left(\omega-\frac{2\pi r}{T}\right),$$

where r is any whole number (including zero as well).

For determining the unknown constant c we take the integral of both parts of the last equality within the limits of $\frac{2\pi r}{T}-\frac{\pi}{T}$ to $\frac{2\pi r}{T}+\frac{\pi}{T}$. Then

$$\int\limits_{\frac{2\pi}{T}\left(r-\frac{1}{2}\right)}^{\frac{2\pi}{T}\left(r+\frac{1}{2}\right)} c\delta\left(\omega-\frac{2\pi r}{T}\right)d\omega = c =$$

$$=\int\limits_{\frac{2\pi}{T}\left(r-\frac{1}{2}\right)}^{\frac{2\pi}{T}\left(r+\frac{1}{2}\right)} \left[1+2\lim_{N\to\infty}\sum_{p=1}^{2N}\left(1-\frac{p}{2N+1}\right)\cos p\omega T\right]d\omega =$$

$$=\frac{2\pi}{T}+4\lim_{N\to\infty}\sum_{p=1}^{2N}\left(1-\frac{p}{2N+1}\right)\frac{\cos rp\pi \sin p\pi}{pT}=\frac{2\pi}{T}.$$

In this manner

$$\lim_{N\to\infty}\Delta_N(\omega)=\frac{2\pi}{T}\delta\left(\omega-\frac{2\pi r}{T}\right). \tag{10.17}$$

Substituting (10.17) into (10.16), we obtain the following general expression of the power spectrum of a pulse random process:

$$F(\omega)=\frac{2}{T}\left\{K(\omega)-|H(\omega)|^2+\psi(\omega)+ + |H(\omega)|^2\cdot\frac{2\pi}{T}\sum_{r=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi r}{T}\right)\right\}, \tag{10.18}$$

in which the functions $K(\omega)$, $H(\omega)$, $\varphi(\omega)$ and $H_p(\omega)$ are determined by formulas (10.8), (10.14), (10.15) and (10.9).

If the pulses are mutually independent, then $H_p(\omega) = |H(\omega)|^2$ and in (10.18) it should be assumed that $\varphi(\omega) \equiv 0$. In this case the pulse distortions (amplitudinal, in duration and in position) are similar to white noise. In some problems the assumption of pulse independence may rest on entirely firm ground. Correlation between pulses may be neglected, if the correlation time $\tau_0 \ll T$. Thus, for instance in examining at the output of a receiver a sequence of video pulses distorted by fluctuation noise, these distortions may be considered to possess the character of white noise if the pass band width of the receiver, $\Delta \gg \frac{1}{\tau}$, where τ is the pulse duration.

Problems may, however, also be encountered in which it will be necessary to take into account the bands of the previous stages, i.e., correlation between the pulses.

The general expression (10.18) of the power spectrum of a pulse random process consists of a continuous part

$$F_{н}(\omega)=\frac{2}{T}\{K(\omega)-|H(\omega)|^2+\psi(\omega)\} \tag{10.19}$$

and a discrete part

$$F_{д}(\omega)=\frac{4\pi}{T^2}|H(\omega)|^2\sum_{r=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi r}{T}\right), \tag{10.20}$$

which consists of discrete lines on frequencies which are multiples of the mean frequency $\frac{2\pi}{T}$ of pulse repetition. The ratio of the full power of the components of the continuous spectrum to the full power of the components of the discrete spectrum is equal to

$$\mu=\frac{\int_{-\infty}^{\infty}[K(\omega)-|H(\omega)|^2+\psi(\omega)]\,d\omega}{\frac{2\pi}{T}\sum_{r=-\infty}^{\infty}\left|H\left(\frac{2\pi r}{T}\right)\right|^2}. \tag{10.21}$$

It is not difficult to see that if distortions are absent, then $K(\omega)=|H(\omega)|^2=|g(\omega)|^2$, where $g(\omega)$ is the spectrum density of the undistorted pulse. Here the continuous part of the spectrum disappears, and the discrete part coincides with the power spectrum of the periodic sequence of unmodulated pulses. Therefore, in those cases where the distortions are caused by interference, it is valid to identify the continuous part of spectrum (10.19) with the interference spectrum, and the discrete part (10.20) with the spectrum of the useful signal. In this case the ratio μ, computed according to (10.21), will yield the ratio of the power of the interference to the power of the signal $\left(\frac{\Pi}{C}\right)$. In other problems where the useful modulation of the pulse sequence is of a statistical character (for instance the modulation of speech in multichannel telephony), the continuous part of the spectrum carries useful information.

If the pulse parameters (amplitude, time of emergence, duration) are, besides purely random distortions, subjected to modulation by a given periodic function of time, then the discrete part (10.20) of the power spectrum must be supplemented by terms containing delta-functions at frequencies corresponding to the components of the resolution of the indicated periodic function into a Fourier series.

In the succeeding sections there are examined some special cases of random

pulse processes, for which the random factor is one of the parameters characterizing the shape of the pulse or the instant of its emergence. Formula (10.18) also permits an investigation of more general cases, when the indicated random distortions act simultaneously.

2. Sequence of Equidistant Pulses having Equal Width and Random Amplitude

Let us examine a sequence of equidistant pulses of given shape, which are of equal width and of random amplitude.

Let us designate by $g(\omega)$ the spectrum density of a pulse with an amplitude equal to unity, which emerges at the instant of time t = 0. Let us designate by T the period of repetition of the pulses, and by ξ_n the random amplitude of the n-th pulse (Fig. 76).

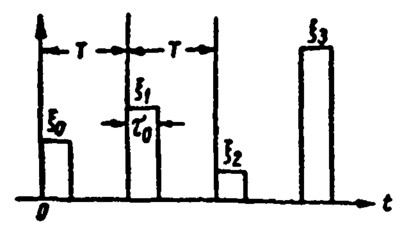


Fig. 76. Sequence of equidistant noises of equal width and with random amplitude.

Since in the case at hand $v_n^{(k)} \equiv 0$ and $F_n(\omega) = \xi_n g(\omega)$, it follows from (10.2) that

$$V_n = \xi_n g(\omega). \tag{10.22}$$

Substituting (10.22) into (10.8) and (10.9), we obtain

$$K(\omega) = m_1\left\{\xi_n^2 |g(\omega)|^2\right\} = |g(\omega)|^2 m_1\left\{\xi_n^2\right\}, \tag{10.23}$$

$$H_{n-j}(\omega) = m_1\left\{\xi_n \xi_j |g(\omega)|^2\right\} = |g(\omega)|^2 m_1\left\{\xi_n \xi_j\right\}. \tag{10.24}$$

Let $w_1(x)$ be the one-dimensional distribution function of the random amplitude ξ_n, equal for all pulses, i.e., for any n. The mean value of a and the dispersion

σ^2 of the random amplitudes of the pulses are equal to

$$m_1\{\xi_n\}=a=\int_{-\infty}^{\infty} x w_1(x)\,dx,$$
$$M_2\{\xi_n\}=\sigma^2=\int_{-\infty}^{\infty}(x-a)^2 w_1(x)\,dx. \tag{10.25}$$

Let $w_2(x,y,\tau)$ be the two-dimensional distribution function of the random amplitudes ξ_n, which depends only on the relative position of the pulses $\tau=(n-j)T$, i.e., only on the difference in the numerals of these pulses. We designate by $R_{n-j}=R[(n-j)T]$ the correlation coefficient of the random amplitudes of the pulses; then

$$m_1\{(\xi_n-a)(\xi_j-a)\}=\sigma^2 R_{n-j}=$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-a)(y-a)\,w_2[x,\ y,\ (n-j)T]\,dx\,dy. \tag{10.26}$$

Employing (2.74) and (3.53), it is not difficult now to express $K(\omega)$ and $H_{n-j}(\omega)$ in terms of the numerical characteristics of the random amplitudes of the pulses

$$K(\omega)=|g(\omega)|^2(\sigma^2+a^2), \tag{10.27}$$
$$H_{n-j}(\omega)=|g(\omega)|^2(\sigma^2 R_{n-j}+a^2). \tag{10.28}$$

If the amplitudes of any pulse pair are independent, then

$$R_p\equiv 0,\quad p\neq 0, \tag{10.29}$$

and from (10.13) and (10.28) it follows that

$$H(\omega)\equiv a\,g(\omega). \tag{10.30}$$

Substituting (10.27), (10.28) and (10.30) into (10.18), we obtain the final expression for the power spectrum of the pulse random process under consideration:

$$F(\omega)=\frac{2}{T}|g(\omega)|^2\left\{\sigma^2[1+\psi_1(\omega)]+\right.$$
$$\left.+\frac{2\pi}{T}a^2\sum_{r=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi r}{T}\right)\right\}. \tag{10.31}$$

where by $\psi_1(\omega)$ is designated

$$\psi_1(\omega)=2\lim_{N\to\infty}\sum_{p=1}^{2N}\left(1-\frac{p}{2N+1}\right)R_p\cos p\omega T. \tag{10.32}$$

It follows from (10.31) that the power spectrum of a sequence of equidistant pulses of random amplitude depends on their correlation function and does not depend on the form of the distribution function of the random amplitudes.

If the random amplitudes of any pair of a sequence of rectangular pulses are independent, then such a pulse process may be regarded as the amplitude modulation of the second kind* of pulses by white noise. Since, when condition (10.29) is fulfilled, $\psi(\omega) \equiv 0$, it follows from (10.31) that for this case we find

$$F(\omega)=\frac{2}{T}|g(\omega)|^2\left\{\sigma^2+\frac{2\pi}{T}a^2\sum_{r=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi r}{T}\right)\right\}. \tag{10.33}$$

The continuous part of power spectrum (10.33) has the same shape as the spectrum of a single pulse, and its intensities are proportional to the dispersion σ^2. The discrete part of this spectrum corresponds to a periodic sequence of pulses of the same shape, but with a constant amplitude equal to the mean value a. Thus with a given pulse shape, the spectrum under examination is determined only by two numerical characteristics, the mean value a and the dispersion σ^2.

The ~~power~~ corresponding to the continuous part of the spectrum is equal to

$$\frac{2\sigma^2}{T}\int_{-\infty}^{\infty}|g(\omega)|^2 d\omega,$$

and that corresponding to its discrete part is

$$\frac{2a^2}{T}\int_{-\infty}^{\infty}\frac{2\pi}{T}|g(\omega)|^2\sum_{r=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi r}{T}\right)d\omega=$$

$$=\frac{2a^2}{T}\sum_{r=-\infty}^{\infty}\left|g\left(\frac{2\pi r}{T}\right)\right|^2\frac{2\pi}{T}\approx\frac{2a^2}{T}\int_{-\infty}^{\infty}|g(\omega)|^2 d\omega.$$

The ratio of the power of the continuous and the discrete parts of the spectrum is equal to

$$\mu=\left(\frac{\sigma}{a}\right)^2, \tag{10.34}$$

i.e., to the square of the ratio of the mean-square value to the mean value of the random amplitude of the pulse.

---

* As is well known, in pulse-amplitude modulation of the second kind, the amplitude of each of the pulses remains constant, equal to that value of the modulating function which corresponds to the leading edge of the given pulse [5].

Figure 77 shows the power spectrum of a sequence of equidistant rectangular pulses with a duration of τ_0, modulated in amplitude by white noise.

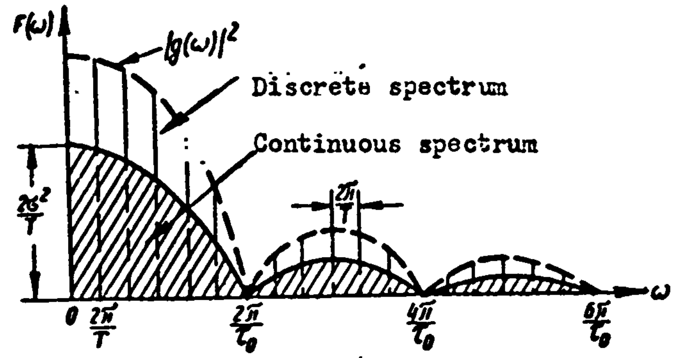


Fig. 77. Power spectrum of a sequence of mutually independent rectangular pulses with random amplitude.

From (10.33) it is not difficult, by means of an inverse Fourier transformation, to find also the correlation function of a sequence of mutually independent rectangular pulses with random amplitude (Fig. 78). Here the discrete part of the spectrum is transformed into a periodic sequence of triangles (Fig. 79a):

$$B_{\text{д}}(\tau)=\frac{a^2}{T^2}\sum_{r=-\infty}^{\infty}\int_{-\infty}^{\infty}|g(\omega)|^2\delta\left(\omega-\frac{2\pi r}{T}\right)e^{i\omega\tau}\,d\omega=$$

$$=\frac{a^2}{T^2}\sum_{r=-\infty}^{\infty}\left|g\left(\frac{2\pi r}{T}\right)\right|^2 e^{\frac{2\pi i r\tau}{T}},$$

wherefrom

$$B_{\text{д}}(\tau)=\begin{cases}\frac{a^2}{T}(\tau_0-|\tau-rT|), & |\tau-rT|<\tau_0,\\ 0, & |\tau-rT|>\tau_0,\end{cases}\qquad (10.35)$$

$$r=0,\ \pm 1,\ \pm 2,\ \ldots$$

The continuous part of the spectrum is transformed into one triangle, located near the point $\tau = 0$ (Fig. 79 b):

$$B_{\text{н}}(\tau)=\frac{\sigma^2}{2\pi T}\int_{-\infty}^{\infty}|g(\omega)|^2 e^{i\omega\tau}\,d\omega=\frac{\sigma^2}{T}(\tau_0-|\tau|),\qquad (10.35')$$

$$|\tau|<\tau_0.$$

Let us return to an examination of the general case, for which the function $\psi_1(\omega)$ is not identically equal to zero.

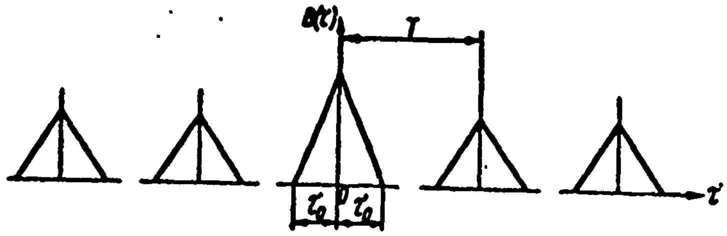


Fig. 78. Correlation function of a sequence of mutually independent rectangular pulses with random amplitude.

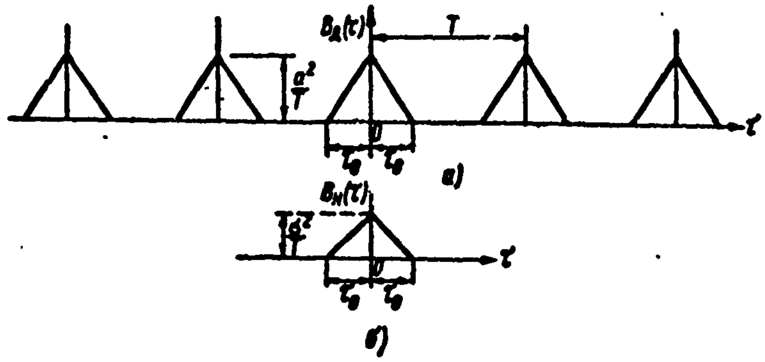


Fig. 79. a) Periodic part of the correlation function shown in Fig. 78; b) Aperiodic part of the correlation function shown in Fig. 78.

If $\sum_{p=1}^{\infty}|R_p|$ converges, then the limit in the right part of (10.32) exists and $\psi_1(\omega)$ is determined by the formula*

$$\psi_1(\omega)=2\sum_{p=1}^{\infty}R_p\cos p\omega T. \tag{10.36}$$

We designate by $\varphi(\omega)$ the Fourier transformation of the correlation coefficient $R(\tau)$, i.e.,

$$\varphi(\omega)=2\int_{-\infty}^{\infty}R(\tau)e^{-i\omega\tau}d\tau=4\int_{0}^{\infty}R(\tau)\cos\omega\tau\,d\tau. \tag{10.37}$$

The function $\varphi(\omega)$ represents the power spectrum of the random process by which the pulse amplitudes are modulated.

Let spectrum $\varphi(\omega)$ have a finite width of 2Δ, i.e., $\varphi(\omega)\equiv 0$ when $|\omega|\geqslant\Delta$.

Let us first assume, that

$$\Delta<\frac{\pi}{T}. \tag{10.38}$$

* If $\xi(kT)$ is a stationary process with discrete time, representing a sequence of random amplitudes, then $\psi_1(\omega)$ is the spectrum density of this process.

Then from (10.36), it follows that $1 + \psi_1(\omega)$ is a periodic function with a period of $\frac{2\pi}{T}$, which within the limits of $\frac{\pi}{T}$ to $\frac{\pi}{T}$ (sic!) coincides with $\varphi(\omega)$. If the inequality (10.38) is not fulfilled, but $\Delta \leqslant \frac{2\pi}{T}$, then, representing Δ as the sum of the two items

$$\Delta = \frac{\pi}{T} + \Delta_1, \quad \Delta_1 < \frac{\pi}{T}, \tag{10.39}$$

we obtain from (10.37)

$$\begin{aligned} R(\tau) = \frac{1}{2\pi}\int_{0}^{\Delta}\varphi(\omega)\, e^{i\omega\tau} d\omega = \frac{1}{\pi}\int_{0}^{\frac{\pi}{T}}\varphi(\omega)\cos\omega\tau\, d\omega + \\ + \frac{1}{\pi}\int_{\frac{\pi}{T}}^{\Delta}\varphi(\omega)\cos\omega\tau\, d\omega = R_1(\tau) + R_2(\tau). \end{aligned} \tag{10.40}$$

Then the series (10.36) may be broken down into a sum of two series:

$$\psi_1(\omega) = \psi_{11}(\omega) + \psi_{12}(\omega), \tag{10.41}$$

where

$$\psi_{11}(\omega) = 2\sum_{p=1}^{\infty} R_{1_p}\cos p\omega T, \tag{10.42}$$

$$\psi_{12}(\omega) = 2\sum_{p=1}^{\infty} R_{2_p}\cos p\omega T. \tag{10.43}$$

The functions $\psi_{11}(\omega)$ and $\psi_{12}(\omega)$ are periodic functions, which (to an accuracy of a direct component, equal to unity) coincide with the function $\varphi(\omega)$: the first in the sector to $\frac{\pi}{T}$, and the second in the sector from $\frac{\pi}{T}$ to Δ. Here the second periodic function is shifted with respect to the first by half of the period $\frac{\pi}{T}$. The correlation function, corresponding to this part of the spectrum, represents a sum of the delta-functions $\delta(\tau - pT)$ with an intensity which diminishes with the growth of the numeral p.

With the arbitrary quantity Δ we act in a completely analogous manner, representing the correlation coefficient in the form of a sum corresponding to the break-up of integral (10.40) into sectors which are multiples of $\frac{\pi}{T}$.

Let us examine as an example the continuous part of the power spectrum of a

sequence of equidistant rectangular pulses, modulated in amplitude by noise with a spectrum uniform in the limited band Δ. In this case the correlation coefficient of the random amplitudes of any pair of pulses is equal to

$$R_p = \frac{\sin p\Delta T}{p\Delta T}. \tag{10.44}$$

If the width of band Δ satisfies the inequality (10.38), then the examined power spectrum will have the form of periodically recurring bands with a width of 2Δ, at frequencies which are multiples of $\frac{2\pi}{T}$ (i.e., of the frequency of pulse recurrence), limited underneath by the abscissa and above by the curve $|g(\omega)|^2$ (Fig. 80). As Δ is increased the width of these bands increases, and when $\Delta = \frac{\pi}{T}$ the gaps between the cadence frequencies disappear. In this case the power spectrum does not differ from a spectrum with an endless band of modulating noise (Fig. 77).

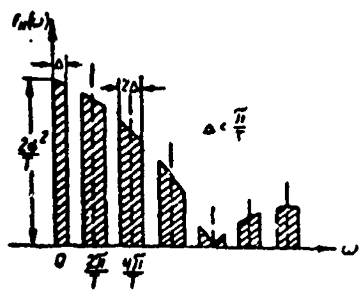


Fig. 80. Power spectrum of sequence of rectangular pulses, modulated in amplitude by noise occupying a limited band of frequencies.

Generally with a band width of Δ, which is a multiple of half the pulse recurrence frequency, as can be seen from (10.44), the correlation coefficient turns to zero ($p \neq 0$), and, consequently, $\psi_1(\omega) \equiv 0$, i.e., the power spectrum must coincide with the spectrum corresponding to an endless band of modulating noise. However, it does not follow from this that the amplitudes of any pulse pair must be independent. In this present case we have still another example of the fact that two quantities, for which $R = 0$, are not necessarily independent (cf. p. 73).

It can also be seen from (10.44) that, as the width of the modulating noise band is increased, the maxima of the quantities R_p for a given p diminish in inverse proportion to Δ, as a result of which the deformation of the envelope of the continuous

spectrum when $\Delta \to \infty$ becomes constantly less noticeable. With $\Delta T \gg 1$ it is virtually possible to consider that this spectrum does not differ from a spectrum corresponding to modulation by white noise.

3. Sequence of Pulses having Equal Amplitudes and Durations, but a Random Emergence Time.

Let us examine a sequence of pulses of given shape which have equal amplitude and duration, but a random time of emergence (Fig. 81).

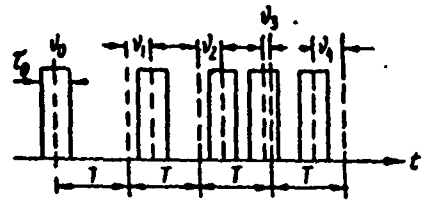


Fig. 81. Sequence of pulses having equal amplitudes and duration, but a random emergence time.

Let $g(\omega)$ be the spectrum density of a pulse emerging at the time instant, t = 0. Since in the case at hand $F_n(\omega) = g(\omega)$, it therefore follows from (10.2) that

$$V_n = g(\omega) e^{i\omega\nu_n}. \tag{10.45}$$

Substituting (10.45) into (10.8) and (10.9), we obtain

$$K(\omega) = m_1\{|g(\omega)|^2\} = |g(\omega)|^2, \tag{10.46}$$

$$H_{n-j}(\omega) = m_1\{|g(\omega)|^2 e^{i\omega(\nu_n - \nu_j)}\} = |g(\omega)|^2 m_1\{e^{i\omega(\nu_n - \nu_j)}\}. \tag{10.47}$$

Let $w_1(x)$ be the one-dimensional distribution function of the random variables ν_n, equal for all pulses, i.e., for any n, and let $w_2(x,y,\tau)$ be the two-dimensional distribution function of these random variables, depending only on the relative position of the pulses, i.e., on the difference n - j of the pulse numerals.

We also introduce the characteristic function $\Theta_2(\omega_1, \omega_2, \tau)$ of the two-dimensional law of distribution

$$\Theta_2(\omega_1, \omega_2, \tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w_2(x, y, \tau) e^{i(\omega_1 x + \omega_2 y)} dx dy,$$

with the aid of which the expression (10.47) can be rewritten in the form of

$$H_{n-j}(\omega)=|g(\omega)|^2\Theta_2[\omega,\ -\omega,\ (n-j)T]. \tag{10.48}$$

If the moments of emergence of any pair of pulses are independent, then

$$H_{n-j}(\omega)=|H(\omega)|^2=|g(\omega)|^2|\Theta_1(\omega)|^2, \tag{10.49}$$

where $\Theta_1(\omega)$ is the characteristic function corresponding to the one-dimensional distribution $w_1(x)$.

We designate by $\psi_2(\omega)$ the limit

$$\psi_2(\omega)=2\lim_{N\to\infty}\sum_{p=1}^{2N}\left(1-\frac{p}{2N+1}\right)\{\Theta_2[\omega,-\omega,(n-j)T]-|\Theta_1(\omega)|^2\}\cos p\omega T. \tag{10.50}$$

Substituting (10.46) and (10.49) into (10.18) and employing (10.50), we find the final expression for the power spectrum of the pulse random process under examination

$$F(\omega)=\frac{2}{T}|g(\omega)|^2\left\{1-|\Theta_1(\omega)|^2+\psi_2(\omega)+\right.$$
$$\left.+\frac{2\pi}{T}|\Theta_1(\omega)|^2\sum_{r=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi r}{T}\right)\right\}. \tag{10.51}$$

Distinct from spectrum (10.31), the power spectrum (10.51) depends on the characteristic functions of the distribution laws of random deviations from the pulse emergence time nT, the square of the modulus of the pulse spectrum density serving, as in (10.31), as a proportionality factor.

When the random moments of emergence of any pair of pulses are independent, which corresponds to time modulation of the second kind* by white noise, then

$$\Theta_2(\omega,\ -\omega,\ pT)=|\Theta_1(\omega)|^2,$$

and from (10.50) it follows that $\psi_2(\omega)\equiv 0$.

From (10.51) for this case we have

$$F(\omega)=\frac{2}{T}|g(\omega)|^2\left\{1-|\Theta_1(\omega)|^2+\right.$$
$$\left.+\frac{2\pi}{T}|\Theta_1(\omega)|^2\sum_{r=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi r}{T}\right)\right\}. \tag{10.52}$$

* As is well known, with pulse-time modulation of the second kind, the position of each of the pulses in a given cadence period is determined by the value of the modulating function at the beginning of this cadence period [5].

From (10.52) it can be seen that the intensities of the discrete part of the power spectrum, with time modulation of the second kind of the pulses by white noise, are proportional to $|\Theta_1(\omega)|^2$, and the intensities of the continuous part of the spectrum are proportional to $1 - |\Theta_1(\omega)|^2$, i.e., in sum these intensities are equal to the square of the modulus of the spectrum density of a pulse (the power of a single pulse).

The power of the process corresponding to the continuous part of the power spectrum consists of

$$\frac{2}{T}\int_{-\infty}^{\infty}|g(\omega)|^2[1-|\Theta_1(\omega)|^2]\,d\omega,$$

and that corresponding to its discrete part is approximately equal to

$$\frac{2}{T}\int_{-\infty}^{\infty}|g(\omega)|^2|\Theta_1(\omega)|^2\,d\omega.$$

4. Examples

Let us examine several specific examples of sequences of pulses with equal amplitudes and durations, but which emerge at random moments in time.

A. Modulating noise—normal

Let the deviations of the emergence time of the pulses from the mean value constitute a normal stationary random process* (modulating noise) with a zero mean value and a correlation coefficient of $R(\tau)$. The one-dimensional and two-dimensional characteristic functions of the process are equal to [cf. (3.79) and (3.95)]

$$\Theta_1(\omega)=e^{-\frac{\sigma^2\omega^2}{2}},\quad \Theta_2(\omega_1,\ \omega_2,\ \tau)=e^{-\frac{\sigma^2}{2}\left[\omega_1^2+2R(\tau)\,\omega_1\omega_2+\omega_2^2\right]} \tag{10.53}$$

Substituting (10.53) into (10.50), we obtain (under the assumption that $\sum_{p=1}^{\infty}|R_p|$ converges) as expression of the function $\psi_2(\omega)$ in the indicated example

$$\begin{aligned}\psi_2(\omega)&=2\sum_{p=1}^{\infty}\left[e^{-\sigma^2\omega^2(1-R_p)}-e^{-\sigma^2\omega^2}\right]\cos p\omega T=\\&=2e^{-\sigma^2\omega^2}\sum_{p=1}^{\infty}\left(e^{-\sigma^2R_p\omega^2}-1\right)\cos p\omega T,\end{aligned} \tag{10.54}$$

---

* Here, of course, is violated the condition of non-overlapping formulated in #1. However, if the dispersion σ^2 of the modulating noise is small in comparison to T^2, the error due to the violation of the indicated condition will be negligible.

where

$$R_p = R(pT).$$

Resolving the exponential function into a series and changing the order of summation, we find

$$\psi_2(\omega) = e^{-\sigma^2\omega^2} \sum_{k=1}^{\infty} \frac{\sigma^{2k}\omega^{2k}}{k!} 2 \sum_{p=1}^{\infty} R_p^k \cos p\omega T$$

and, introducing the designation

$$\psi_{2k}(\omega) = 2 \sum_{p=1}^{\infty} R_p^k \cos p\omega T, \tag{10.55}$$

we obtain

$$\psi_2(\omega) = e^{-\sigma^2\omega^2} \sum_{k=1}^{\infty} \frac{\sigma^{2k}\omega^{2k}}{k!} \psi_{2k}(\omega). \tag{10.56}$$

We designate by $\varphi_K(\omega)$ the Fourier transformation of the k-th degree of the correlation coefficient

$$\varphi_k(\omega) = \int_{-\infty}^{\infty} R^k(\tau) e^{-i\omega\tau} d\tau = 2\int_0^{\infty} R^k(\tau) \cos \omega\tau \, d\tau. \tag{10.57}$$

Then under conditions analogous to (10.38) and (10.39), expression (10.55) represents a Fourier series (or a sum of Fourier series) of a periodic function with a period of $\frac{2\pi}{T}$, coinciding within the limits of one period with $\varphi_K(\omega)$. If the spectrum of the modulating noise is uniform in the limited band Δ, and, consequently, the correlation coefficient satisfies the relationship (10.44), then $\psi_{21}(\omega)$, $\psi_{22}(\omega)$ represent respectively a periodic step-shaped and a periodic saw-tooth function.

When the width of band Δ of the modulating noise is a multiple of half the pulse recurrence frequency, $R_p = 0$, as can be seen from (10.53), $\psi_2(\omega) \equiv 0$. In this case the power spectrum coincides with the spectrum corresponding to an endless band of modulating noise, and has the form of

$$F(\omega) = \frac{2}{T} |g(\omega)|^2 \Big\{1 - e^{-\sigma^2\omega^2} + \frac{2\pi}{T} e^{-\sigma^2\omega^2} \sum_{r=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi r}{T}\right)\Big\}. \tag{10.58}$$

For rectangular pulses with a height of u_0 and a duration of τ_0, the spectrum density is $g(\omega) = \frac{2u_0}{\omega} \sin \frac{\omega\tau_0}{2}$ and the power spectrum is equal to

$$F(\omega) = \frac{8u_0^2}{\omega^2 T} \sin^2 \frac{\omega\tau_0}{2} \left\{ 1 - e^{-\sigma^2\omega^2} + \frac{2\pi}{T} e^{-\sigma^2\omega^2} \sum_{r=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi r}{T}\right) \right\}. \qquad (10.58')$$

The corresponding correlation function has a periodic part:

$$B_n(\tau) = \frac{1}{4\pi} \sum_{r=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{8u_0^2}{\omega^2 T} \sin^2 \frac{\omega\tau_0}{2} \frac{2\pi}{T} e^{-\sigma^2\omega^2} \delta\left(\omega - \frac{2\pi r}{T}\right) e^{i\omega\tau} d\omega =$$

$$= \frac{2u_0^2}{\pi^2} \sum_{r=1}^{\infty} a_r \cos \frac{2\pi r\tau}{T},$$

where

$$a_r = \frac{1}{r^2} e^{-4\pi^2 r^2 \left(\frac{\sigma}{T}\right)^2} \sin^2 \frac{\pi r \tau_0}{T},$$

and an aperiodic part

$$B_H(\tau) = \frac{1}{2\pi} \cdot \frac{8u_0^2}{T} \int_0^{\infty} \frac{\sin^2 \frac{\omega\tau_0}{2}}{\omega^2} (1 - e^{-\sigma^2\omega^2}) \cos \omega\tau d\omega.$$

The expression for $B_H(\tau)$ represents a sum of two integrals, the first of which is equal to

$$\frac{1}{2\pi} \cdot \frac{8u_0^2}{T} \int_0^{\infty} \frac{\sin^2 \frac{\omega\tau_0}{2}}{\omega^2} \cos \omega\tau d\omega = \frac{1}{2\pi} \frac{8u_0^2}{T} \cdot \frac{\pi\tau_0}{4} \left(1 - \frac{|\tau|}{\tau_0}\right),$$

and the second, through integration by parts, is reduced to the sum of tabular integrals

$$\frac{1}{2\pi} \frac{8u_0^2}{T} \int_0^{\infty} \frac{\sin^2 \frac{\omega\tau_0}{2}}{\omega^2} e^{-\sigma^2\omega^2} \cos \omega\tau d\omega = \frac{1}{2\pi} \cdot \frac{8u_0^2}{T} \cdot \frac{\tau_0}{4} \Big\{ k_1(|\tau| + \tau_0) -$$

$$- k_1(|\tau| - \tau_0) + \frac{|\tau|}{\tau_0} [k_1(|\tau| + \tau_0) - 2k_1(|\tau|) + k_1(|\tau| - \tau_0)] +$$

$$+ \frac{2\sigma^2}{\tau_0^2} [k_2(|\tau| + \tau_0) + k_2(|\tau| - \tau_0) - 2k_2(|\tau|)] \Big\},$$

where

$$k_1(t) = \int_0^{\infty} \frac{\sin \omega t}{\omega} e^{-\sigma^2\omega^2} d\omega = \frac{\pi}{2} \Phi\left(\frac{t}{2\sigma}\right), \quad t > 0, \quad \sigma > 0,$$

$$k_2(t) = \int_0^{\infty} e^{-\sigma^2\omega^2} \cos \omega t d\omega = \frac{\sqrt{\pi}}{2\sigma} e^{-\frac{t^2}{4\sigma^2}}, \quad t > 0, \quad \sigma > 0,$$

and $\Phi(x)$ is a Kramp function (cf. p. 26).

Figures 82 and 83 show the power spectrum and correlation function of a periodic sequence of rectangular pulses with a constant amplitude of u_0 and a duration of τ_0, whose instant of emergence is distorted by white normal noise with a dispersion of $\sigma^2 \ll \tau_0^2$.

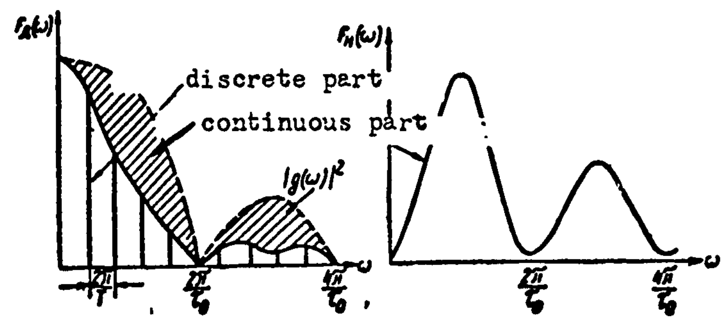


Fig. 82. Power spectrum of a sequence of mutually independent rectangular pulses emerging at a random instant in time.

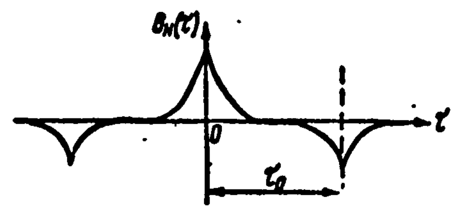


Fig. 83. Aperiodic part of the correlation function of a sequence of mutually independent rectangular pulses, emerging at a random instant in time.

The power of the discrete part of spectrum (10.58') in the case under examination is equal approximately to

$$\frac{8u_0^2}{\pi^2T^2}\int_{-\infty}^{\infty}\frac{\sin^2\frac{\omega\tau_0}{2}}{\omega^2}e^{-\sigma^2\omega^2}d\omega=\frac{4u_0^2\tau_0}{\pi T}\left[\Phi\left(\frac{\tau_0}{2\sigma}\right)-\frac{2\sigma}{\tau_0\sqrt{\pi}}\left(1-e^{-\frac{\tau_0^2}{4\sigma^2}}\right)\right],$$

and the power of the continuous part of this spectrum is

$$\frac{8u_0^2}{\pi^2T^2}\int_{-\infty}^{\infty}\frac{\sin^2\frac{\omega\tau_0}{2}}{\omega^2}\left(1-e^{-\sigma^2\omega^2}\right)d\omega=$$

$$=\frac{4u_0^2\tau_0}{\pi T}\left[1-\Phi\left(\frac{\tau_0}{2\sigma}\right)+\frac{2\sigma}{\tau_0\sqrt{\pi}}\left(1-e^{-\frac{\tau_0^2}{4\sigma^2}}\right)\right].$$

When $\tau_0 \gg \sigma$, employing an asymptotic resolution for $\Phi(x)$ [cf. (1.40) and (2.20)], we obtain

$$\mu\approx\frac{2\sigma}{\tau_0\sqrt{\pi}}.$$

B. Exponential distribution of the modulating noise.

Let the one-dimensional and two-dimensional distribution functions of the time intervals between pulses be given by expressions (8.20') and (8.19)*. The characteristic functions corresponding to them are determined by formula (8.24). It follows from this formula that

$$\Theta_2(\omega, -\omega) = \frac{1}{1 + 4\sigma^4\omega^2(1 - R^2)}, \quad \Theta_1(\omega) = \frac{1}{1 - 2i\sigma^2\omega}.$$

Then from (10.50) (under the assumption that $\sum_{p=1}^{\infty} |R_p|$ converges), we obtain the following expression of function $\psi_2(\omega)$

$$\psi_2(\omega) = \frac{8\sigma^4\omega^2}{1 + 4\sigma^4\omega^2} \sum_{p=1}^{\infty} \frac{R_p^2}{1 + 4\sigma^4\omega^2(1 - R_p^2)} \cos p\omega T.$$

In accordance with (10.51) the power spectrum of the pulse random process under examination has the form of

$$F(\omega) = \frac{2}{T}|g(\omega)|^2 \left\{ \frac{4\sigma^4\omega^2}{1 + 4\sigma^4\omega^2} \left[1 + 2\sum_{p=1}^{\infty} \frac{R_p^2}{1 + 4\sigma^4\omega^2(1 - R_p^2)} \times \right.\right.$$
$$\left.\left. \times \cos p\omega T \right] + \frac{2\pi}{T} \frac{1}{1 + 4\sigma^4\omega^2} \sum_{r=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi r}{T}\right) \right\}. \quad (10.59) \tag{10.59}$$

If the spectrum of the modulating noise is uniform in band Δ, which is a multiple of half the pulse recurrence frequency, then $R_p = 0 (p \geqslant 1)$. Then the power spectrum coincides with the spectrum corresponding to modulation by white noise, and from (10.59) we find

$$F(\omega) = \frac{2}{T}|g(\omega)|^2 \left\{ \frac{4\sigma^4\omega^2}{1 + 4\sigma^4\omega^2} + \frac{2\pi}{T} \cdot \frac{1}{1 + 4\sigma^4\omega^2} \sum_{r=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi r}{T}\right) \right\}. \tag{10.59'}$$

C. Other forms of distribution of white noise

Let us examine two more examples of the position modulation of pulses by white noise. Let the noise distribution be uniform over the interval from $-\alpha T$ to αT

---

* Let us note, that if in the exponential distribution (8.20') the substitution $\nu t = \frac{\rho}{2\sigma^2}$ is made, we obtain $w(\nu t) = e^{-\nu t}$, i.e., the probability that a random variable distributed according to Poisson's law (1.54) will not emerge during the period of time t. Cf. also note on p. 393 .

$\left(0<\alpha<\frac{1}{2}-\frac{\tau_0}{T}\right)$. The corresponding characteristic function, according to (3.85), is equal to

$$\Theta_1(\omega)=\frac{\sin\alpha\omega T}{\alpha\omega T}.$$

The power spectrum in this case has the form of

$$F(\omega)=\frac{8u_0^2}{\omega^2 T}\sin^2\frac{\omega\tau_0}{2}\left[1-\frac{\sin^2\alpha\omega T}{\alpha^2\omega^2T^2}+\right.$$
$$\left.+\frac{2\pi}{T}\,\frac{\sin^2\alpha\omega T}{\alpha^2\omega^2T^2}\sum_{r=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi r}{T}\right)\right]. \qquad (10.60)$$

If the noise distribution is equal to [cf. (3.16)]

$$w_1(x)=\frac{1}{\pi\alpha T}\cdot\frac{1}{\sqrt{1-\frac{x^2}{\alpha^2T^2}}},\quad |x|<\alpha T,$$

then the corresponding characteristic function, according to (3.90), is

$$\Theta_1(\omega)=J_0(\alpha\omega T),$$

and the power spectrum is

$$F(\omega)=\frac{8u_0^2}{\omega^2T}\sin^2\frac{\omega\tau_0}{2}\left[1-J_0^2(\alpha\omega T)+\right.$$
$$\left.+\frac{2\pi}{T}J_0^2(\alpha\omega T)\sum_{r=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi r}{T}\right)\right]. \qquad (10.61)$$

5. Sequence of Equidistant Pulses with Equal Amplitude and Random Duration.

Still another type of pulse random process of great practical importance is a sequence of pulses of given shape, which have the same amplitude and random duration.

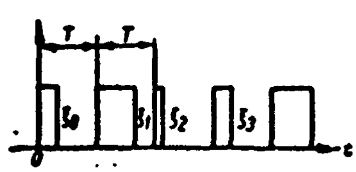


Fig. 84. Sequence of equidistant pulses of equal amplitude and random duration.

Let $g(\omega, \xi_0)$ be the spectral density of a pulse emerging at the moment of time t = 0. We designate by ξ_n the random duration of the n-th pulse (Fig. 84). Then

$$F_n(\omega)=g(\omega,\xi_n). \qquad (10.62)$$

Since the pulses are emerging at constant intervals in time, equal to T, $\nu_\eta\equiv 0$,

and from (10.2) it follows that

$$V_n = F_n(\omega) = g(\omega, \xi_n). \tag{10.63}$$

Substituting (10.63) into (10.8) and (10.9), we obtain

$$K(\omega) = m_1\{|g(\omega, \xi_n)|^2\}, \tag{10.64}$$

$$H_{n-j}(\omega) = m_1\{g(\omega, \xi_n)\overline{g(\omega, \xi_j)}\}. \tag{10.65}$$

Let $W_1(x)$ be the one-dimensional distribution function of the random durations ξ_n, the same for all pulses, i.e., for any n, and let $w_2(x, y, \tau)$ be the two-dimensional distribution function of these random variables, depending only on the relative position of the pulses, i.e., on the difference n - j of the pulse numerals. Then from (10.64) and (10.65) it follows* that

$$K(\omega) = \int_{-\infty}^{\infty} |g(\omega, x)|^2 w_1(x)\,dx, \tag{10.66}$$

$$H_{n-j}(\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\omega, x)\overline{g(\omega, y)} w_2[x, y, (n-j)T]\,dx\,dy. \tag{10.67}$$

If the widths of any pair of pulses are independent, then

$$H_{n-j}(\omega) = \int_{-\infty}^{\infty} g(\omega,x) w_1(x)\,dx \cdot \int_{-\infty}^{\infty} \overline{g(\omega,y)} w_1(y)\,dy = \tag{10.68}$$

$$= \left|\int_{-\infty}^{\infty} g(\omega,x) w_1(x)\,dx\right|^2 = |H(\omega)|^2,$$

where

$$H(\omega) = \int_{-\infty}^{\infty} g(\omega,x) w_1(x)\,dx. \tag{10.68'}$$

We designate by $\psi_3(\omega)$ the limit

$$\psi_3(\omega) = 2\lim_{N\to\infty}\sum_{p=1}^{2N}\left(1-\frac{p}{2N+1}\right)[H_p(\omega) - |H(\omega)|^2]\cos p\omega T. \tag{10.69}$$

Then from the general formula (10.13) there follows the final expression of the power spectrum of the pulse random process under examination

$$F(\omega) = \frac{2}{T}\Big\{K(\omega) - |H(\omega)|^2 + \psi_3(\omega) +$$

$$+ \frac{2\pi}{T}|H(\omega)|^2 \sum_{r=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi r}{T}\right)\Big\}. \tag{10.70}$$

---

* Strictly speaking, the limits of integration should be restricted by the interval (0, T). Here unrestricted integration limits are taken conditionally considering the footnote on p. 3??.

in which the function $K(\omega)$, $H(\omega)$, $\psi_3(\omega)$ and $H_p(\omega)$ are determined by formulas (10.66) - (10.69).

In distinction from spectra (10.31) and (10.51), in power spectrum (10.70) the statistical characteristics of the process and the spectrum density of a single pulse cannot be separated, they enter jointly into function $K(\omega)$ and $H_p(\omega)$.

If the random widths of any pair of pulses are independent, which corresponds to time modulation of the second kind* by white noise, then $H_p(\omega) \equiv |H(\omega)|^2$, and from (10.69) it follows that $\psi_3(\omega) \equiv 0$. Then from (10.70) for this case we have

$$F(\omega)=\frac{2}{T}\left\{K(\omega)-|H(\omega)|^2+\frac{2\pi}{T}|H(\omega)|^2\sum_{r=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi r}{T}\right)\right\}. \quad (10.71)$$

With very small dispersions of random pulse widths, power spectrum (10.70) does not differ in structure from the spectrum of a sequence of equidistant pulses with random amplitude. In fact, with smaller instances of x, the function $g(\omega, x)$ may be resolved into a Taylor series, being limited by the linear term

$$g(\omega, x)=g(\omega, 0)+x\left(\frac{\partial g}{\partial x}\right)_{x=0}. \quad (10.72)$$

Since $g(\omega, 0) \equiv 0$, it follows from (10.67) and (10.72) that

$$H_p(\omega)=\left(\frac{\partial g}{\partial x}\right)_{x=0}\cdot\left(\frac{\partial g}{\partial y}\right)_{y=0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}xyw_2(x, y, pT)\,dx\,dy.$$

Considering that

$$\left(\frac{\partial g}{\partial x}\right)_{x=0}=\left(\frac{\partial g}{\partial y}\right)_{y=0}$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}xyw_2(x, y, pT)\,dx\,dy=B_1(pT),$$

where $b_1(\tau)$ is the correlation function of the random pulse widths, we find

$$H_p(\omega)=\left(\frac{\partial g}{\partial x}\right)_{x=0}^2 B_1(pT), \quad (10.73)$$

---

* As is well known, with pulse-width modulation of the second kind the width of each of the pulses is equal to the value of a modulating function corresponding to the leading (or trailing) edge of the given pulse [5].

from which it always follows that

$$|H(\omega)|^2=\left(\frac{\partial g}{\partial x}\right)^2_{x=0} B_1(\infty)=\left(\frac{\partial g}{\partial x}\right)^2_{x=0}\tau_0^2, \tag{10.74}$$

$$K(\omega)=\left(\frac{\partial g}{\partial x}\right)^2_{x=0} B_1(0)=\left(\frac{\partial g}{\partial x}\right)^2_{x=0}(\sigma^2+\tau_0^2), \tag{10.75}$$

where τ_0 and σ^2 are the mean value and the dispersion of the random pulse width.

Substituting (10.73) - (10.75) into (10.70), we find

$$F(\omega)=\frac{2}{T}\left(\frac{\partial g}{\partial x}\right)^2_{x=0}\left\{\sigma^2+\sigma^2\lim_{N\to\infty}2\sum_{p=1}^{2N}\left(1-\frac{p}{2N+1}\right)R_p\cos p\omega T+\right. \tag{10.76}$$

$$\left.+\frac{2\pi}{T}\tau_0^2\sum_{r=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi r}{T}\right)\right\},$$

where

$$R_p=\frac{B_1(pT)-\tau_0^2}{\sigma^2}. \tag{10.76'}$$

Comparing (10.76) with (10.31), we become convinced that the power spectra of pulse sequences, modulated in amplitude and width, have the same form with small temporal deviation.

6. Unilateral Modulation of Rectangular Pulses in Width

Let rectangular pulses with a height of u_0 emerge periodically at intervals of time T, and let the width modulation take place as the result of random shifting of the trailing edge of the pulse. Let us designate by τ_0 the width of an unmodulated pulse and by $w_2(x, y, \tau)$ the distribution function of the random deviation from τ_0. The function $g(\omega, x)$ in the case under examination will be equal to

$$g(\omega, x)=u_0\int_0^{\tau_0+x}e^{i\omega t}dt=\frac{u_0}{i\omega}(e^{i\omega\tau_0}e^{i\omega x}-1). \tag{10.77}$$

Substituting (10.77) into (10.67), we find

$$H_p(\omega)=\frac{u_0^2}{\omega^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(e^{i\omega\tau_0}e^{i\omega x}-1)\times$$

$$\times(e^{-i\omega\tau_0}e^{-i\omega y}-1)w_2(x, y, pT)\,dx\,dy=$$

$$=\frac{u_0^2}{\omega^2}\Big(1+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i\omega(x-y)}w_2(x, y, pT)\,dx\,dy-$$

$$-e^{i\omega\tau_0}\int_{-\infty}^{\infty}e^{i\omega x}w_1(x)\,dx-e^{-i\omega\tau_0}\int_{-\infty}^{\infty}e^{-i\omega y}w_1(y)\,dy,$$

and, introducing the characteristic functions of the random width deviation, we obtain

$$H_p(\omega)=\frac{u_0^2}{\omega^2}[1+\Theta_2(\omega,-\omega,pT)-e^{i\omega\tau_0}\Theta_1(\omega)-e^{-i\omega\tau}\Theta_1(-\omega)]. \tag{10.78}$$

From (10.78), directing $p\to\infty$, we obtain $|H(\omega)|^2$, and with $p = 0$ we obtain $K(\omega)$:

$$|H(\omega)|^2=\frac{u_0^2}{\omega^2}[1+|\Theta_1(\omega)|^2-e^{i\omega\tau_0}\Theta_1(\omega)-e^{-i\omega\tau_0}\Theta_1(-\omega)], \tag{10.78'}$$

$$K(\omega)=\frac{u_0^2}{\omega^2}[2-e^{i\omega\tau_0}\Theta_1(\omega)-e^{-i\omega\tau_0}\Theta_1(-\omega)], \tag{10.78''}$$

$$K(\omega)-|H(\omega)|^2=\frac{u_0^2}{\omega^2}[1-|\Theta_1(\omega)|^2]. \tag{10.78'''}$$

With modulation by white noise the continuous part of the spectrum has the form

$$F_n(\omega)=\frac{2u_0^2}{\omega^2T}[1-|\Theta_1(\omega)|^2], \tag{10.79}$$

and the discrete part

$$F_\partial(\omega)=\frac{4\pi u_0^2}{\omega^2T^2}[1+|\Theta_1(\omega^2)|^2-e^{i\omega\tau_0}\Theta_1(\omega)-$$
$$-e^{-i\omega\tau_0}\Theta_1(-\omega)]\sum_{r=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi r}{T}\right). \tag{10.79'}$$

If the random deviations of pulse width from τ_0 are subject to the normal law* with a zero mean, a dispersion of σ^2 and a correlation coefficient of $R(\tau)$, then, considering (10.53), we obtain from (10.78)

$$H_p(\omega)=\frac{u_0^2}{\omega^2}\left(1+e^{-\sigma^2\omega^2(1-R_p)}-e^{i\omega\tau_0}e^{-\frac{\sigma^2\omega^2}{2}}-e^{-i\omega\tau_0}e^{-\frac{\sigma^2\omega^2}{2}}\right),$$

or

$$H_p(\omega)=\frac{u_0^2}{\omega^2}\left(1+e^{-\sigma^2\omega^2(1-R_p)}-2e^{-\frac{\sigma^2\omega^2}{2}}\cos\omega\tau_0\right). \tag{10.80}$$

From (10.78') and (10.78'') we also find

$$|H(\omega)|^2=\frac{u_0^2}{\omega^2}\left(1+e^{-\sigma^2\omega^2}-2e^{-\frac{\sigma^2\omega^2}{2}}\cos\omega\tau_0\right), \tag{10.81}$$

$$K(\omega)=\frac{2u_0^2}{\omega^2}\left(1-e^{-\frac{\sigma^2\omega^2}{2}}\cos\omega\tau_0\right). \tag{10.81'}$$

---

* Cf. footnote p. 392 .

Substituting (10.80) and (10.81) into (10.69), we obtain (under the assumption that $\sum_{p=1}^{\infty}|R_p|$ converges) an expression for the function

$$\psi_3(\omega)=\frac{2u_0^2}{\omega^2}e^{-\sigma^2\omega^2}\sum_{p=1}^{\infty}(e^{R_p\sigma^2\omega^2}-1)\cos p\omega T.$$

Expanding the exponential function into a series, changing the order of summation and designating

$$\psi_{3k}(\omega)=2\sum_{p=1}^{\infty}R_p^k\cos p\omega T, \tag{10.82}$$

we obtain

$$\psi_3(\omega)=u_0^2 e^{-\sigma^2\omega^2}\sum_{k=1}^{\infty}\frac{\sigma^{2k}\omega^{2k-2}}{k!}\psi_{3k}(\omega). \tag{10.83}$$

The functions $\psi_{3k}(\omega)$ do not differ from the function $\psi_{2k}(\omega)$ examined above [cf. (10.55)].

Substituting (10.81), (10.81') and (10.83) into the general formula (10.70), we obtain the power spectrum of a sequence of rectangular pulses with unilateral width modulation by normal noise

$$\begin{gathered}F_1(\omega)=\frac{2u_0^2}{\omega^2 T}\Big\{1-e^{-\sigma^2\omega^2}+e^{-\sigma^2\omega^2}\sum_{k=1}^{\infty}\frac{\sigma^{2k}\omega^{2k}}{k!}\psi_{3k}(\omega)+\\ +\frac{2\pi}{T}\Big(1+e^{-\sigma^2\omega^2}-2e^{-\frac{\sigma^2\omega^2}{2}}\cos\omega\tau_0\Big)\sum_{r=-\infty}^{\infty}\delta\Big(\omega-\frac{2\pi r}{T}\Big)\Big\}.\end{gathered} \tag{10.84}$$

With modulation by white noise the summation along k in (10.84) is absent, since here $\psi_{3k}(\omega)\equiv 0$. Just as in the types of pulse random processes examined above, with modulation by uniform noise with a limited band the power spectrum turns out to be the same as for white noise, if the width of band Δ is a multiple of half the pulse recurrence frequency. Figure 85 and 86 show the power spectrum and the correlation function of the sequence of rectangular pulses with unilateral width modulation by white normal noise with a dispersion of $\sigma^2 \ll \tau_0^2$. As in the pulse random processes examined above, the power spectrum consists of a discrete and a continuous part, and the correlation function correspondingly of a periodic and an aperiodic part. However, these functions have also their characteristic differences. Let us compare, for instance, the correlation functions with amplitude modulation

and with width modulation by white noise (Figs. 78 and 86). With amplitude modulation the width τ_0 of all the pulses is fixed, overlapping of pulses ceases with a displacement of $\tau \geqslant \tau_0$ and therefore the periodic part of the correlation function turns to zero when $|\tau - rT| \geqslant \tau_0$.

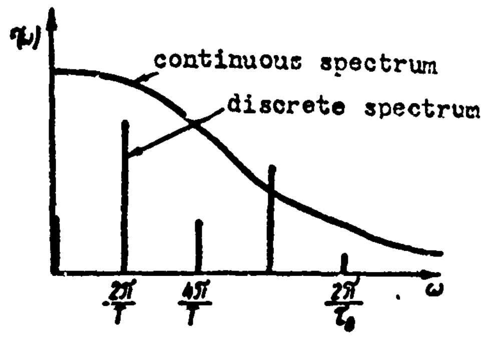


Fig. 85. Power spectrum of sequence of mutually independent rectangular pulses with unilateral random width modulation.

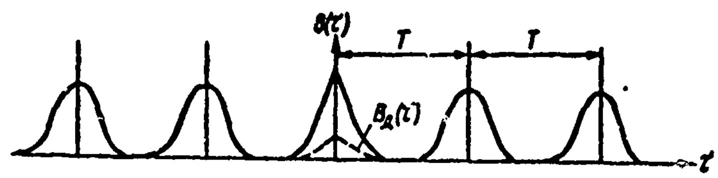


Fig. 86. Correlation function of sequence of mutually independent rectangular pulses with unilateral random width modulation.

With width modulation the trailing edge of the pulses is randomly displaced and therefore with a change in τ the overlapping does not cease simultaneously for all pulse pairs in two realizations of a pulse random process. To this corresponds a rounding of the bottom angles and an expansion of the range of positive values of the correlation function. It can also be seen that the top angle of the triangle in Figure 78 becomes rounded in the periodic part of the correlation function shown in Figure 86. But the aperiodic part retains the angular point when $\tau = 0$.

Employing (3.35) and (3.90) and bearing in mind (10.78') and (10.78"), it is not difficult to write the expression of the power spectrum of a sequence of rectangular pulses with unilateral width modulation by white noise in those cases, where the distribution of this noise is uniform or coincides with the distribution of harmonic

vibration with a random phase. In the first case,

$$F_2(\omega)=\frac{2u_0^2}{\omega^2 T}\left\{1-\frac{\sin^2\alpha\omega T}{\alpha^2\omega^2T^2}+\frac{2\pi}{T}\left(1-2\frac{\sin\alpha\omega T}{\alpha\omega T}\cos\omega\tau_0+\right.\right.$$
$$\left.\left.+\frac{\sin^2\alpha\omega T}{\alpha^2\omega^2T^2}\right)\sum_{r=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi r}{T}\right)\right\}. \tag{10.85}$$

and in the second

$$F_3(\omega)=\frac{2u_0^2}{\omega^2 T}\left\{1-J_0^2(\alpha\omega T)+\frac{2\pi}{T}\left[1-2J_0(\alpha\omega T)\cos\omega\tau_0+\right.\right.$$
$$\left.\left.+J_0^2(\alpha\omega T)\right]\sum_{r=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi r}{T}\right)\right\}. \tag{10.86}$$

In (10.85) and (10.86) the quantity αT is equal to the maximum possible displacement of the trailing edge of the pulse.

From a comparison of (10.84), (10.85) and (10.86) with (10.58'), (10.60) and (10.61), it can be seen that the continuous parts of power spectra of sequences of rectangular pulses with constant amplitude and width, emerging at a random instant in time, differ from the corresponding spectra with unilateral modulation in duration only by the factor $4\sin^2\frac{\omega\tau_0}{2}$ *.

In Figure 87 are superposed the continuous parts of the power spectra of sequences of rectangular pulses with unilateral width modulation by white noise for the three shifts examined above. For comparison (broken curve) there is also included the continuous spectrum of a sequence of rectangular pulses, modulated in amplitude by white noise. All the spectral densities refer to the corresponding values when $\omega = 0$, which are not difficult to determine from (10.84) - (10.86):

$$F_3(0)=\frac{2u_0^2\alpha^2T^2}{2T},\quad F_2(0)=\frac{2u_0^2\alpha^2T^2}{3T},\quad F_1(0)=\frac{2u_0^2\sigma^2}{T}\approx$$
$$\approx\frac{2u_0^2\alpha^2T^2}{9T}. \tag{10.87}$$

(for normal noise the maximum edge displacement is assumed to be $\alpha T \approx 3\sigma$). With amplitude modulation, as can be seen from (10.33), $F(0) = \frac{2u_0^2\sigma^2\tau_0^2}{T}$.

* This conforms fully with (5.63), since a sequence of pulses modulated in position may be regarded as the difference between a pulse sequence with unilateral width modulation and the same sequence retarded by the quantity τ_0 .

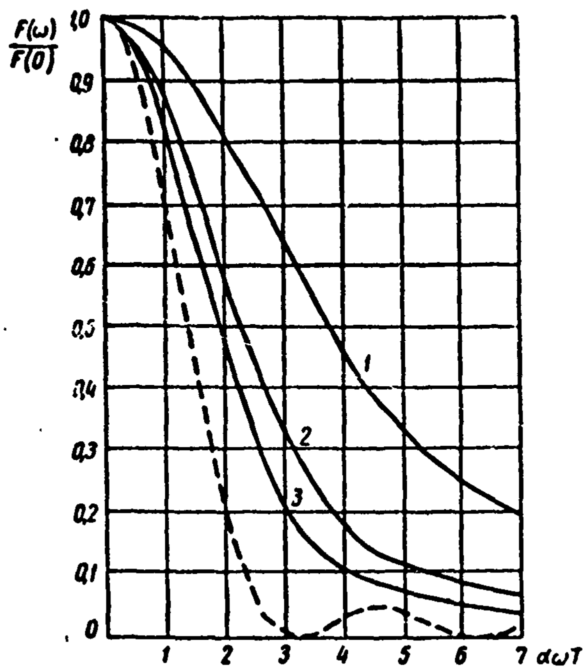


Fig. 87. Comparison of power spectra for three forms of distribution of pulse trailing-edge shifts.

1) normal distribution, 2) uniform distribution

3) distribution of sinusoid with random phase.

Since the spectral densities with $\omega = 0$ characterize the correlation time of a random process, it can ~~therefore be~~ seen from (10.27) that the minimum correlation time corresponds to random edge shifts, distributed according to the normal law.

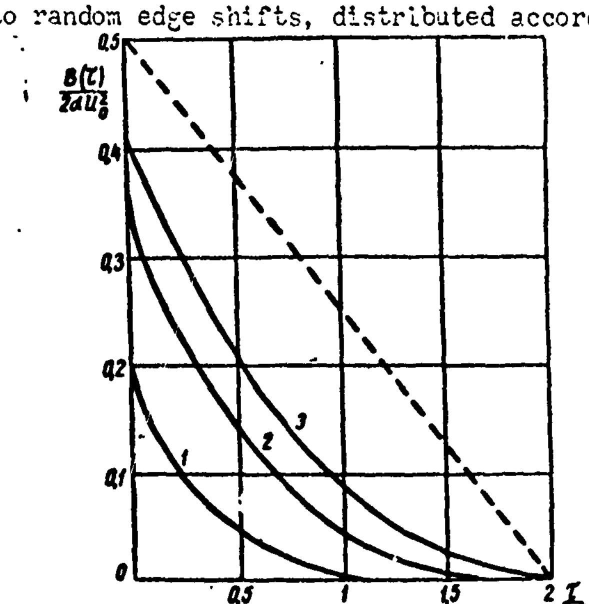


Fig. 88. Comparsion of correlation functions for three forms of distribution of pulse trailing-edge shift.
1) Normal distribution 2) Uniform distribution 3) Distribution of sinusoid with random phase.

This is illustrated by Figure 88, wherein are shown the aperiodic parts of correlation functions of the examined pulse sequences. For comparison (in Fig. 88) is included the aperiodic part of the correlation function of an amplitude-modulated pulse sequence. In this figure are shown the non-dimensional values of the correlation functions, obtained through division by $2u_0^2\alpha$. Let us note that the continuous parts of spectra, as well as the aperiodic parts of correlation functions, do not depend on the duration of the unmodulated pulse.

7. Bilateral Width Modulation of Rectangular Pulses

In the bilateral width modulation of rectangular pulses, it is of interest to examine the following two cases.

A. The leading edge and the trailing edge are shifted independently.

Then

$$g(\omega, x_1, x_2) = u_0 \int_{-\frac{\tau_0}{2}+x_1}^{\frac{\tau_0}{2}+x_2} e^{i\omega t}\,dt = \frac{u_0}{i\omega}\left(e^{\frac{i\omega\tau_0}{2}} e^{i\omega x_2} - e^{-\frac{i\omega\tau_0}{2}} e^{i\omega x_1}\right) = \frac{u_0}{i\omega}\left(e^{\frac{i\omega\tau_0}{2}} e^{i\omega x_2} - 1\right) - \frac{u_0}{i\omega}\left(e^{-\frac{i\omega\tau_0}{2}} e^{i\omega x_1} - 1\right). \tag{10.88}$$

If the statistical characteristics of the shifts of the leading edge and of the trailing edge are the same, then from a comparison of (10.88) with (10.77) we become convinced that, in the case under consideration, the continuous parts of the power spectra and the aperiodic parts of the correlation functions will differ from the corresponding spectra and correlation functions for unilateral modulation only by a constant factor, equal to two.

B. The trailing and leading edges of the pulse are shifted by the same amount, but in opposite directions (symmetrical bilateral width modulation). In this case

$$g(\omega, x) = u_0 \int_{-\frac{\tau_0}{2}-x}^{\frac{\tau_0}{2}+x} e^{i\omega t}\,dt = \frac{u_0}{i\omega}\left(e^{\frac{i\omega\tau_0}{2}} e^{i\omega x} - e^{-\frac{i\omega\tau_0}{2}} e^{-i\omega x}\right), \tag{10.89}$$

and from (10.67) we obtain

$$H_p(\omega)=\frac{u_0^2}{\omega^2}[\Theta_2(\omega,-\omega,pT)+\Theta_2(-\omega,\omega,pT)- \\ -e^{i\omega\tau_0}\Theta_2(\omega,\omega,pT)-e^{-i\omega\tau_0}\Theta_2(-\omega,-\omega,pT)], \quad (10.90)$$

$$|H(\omega)|^2=\frac{u_0^2}{\omega^2}[2|\Theta_1(\omega)|^2-\Theta_1^2(\omega)e^{i\omega\tau_0}-\Theta_1^2(-\omega)e^{-i\omega\tau_0}], \quad (10.90')$$

$$K(\omega)=\frac{u_0^2}{\omega^2}[2-\Theta_1(2\omega)e^{i\omega\tau_0}-\Theta_1(-2\omega)e^{-i\omega\tau_0}]. \quad (10.90'')$$

With modulation by white noise the continuous part of the spectrum has the form

$$F_{\text{н}}(\omega)=\frac{2}{T}[K(\omega)]-|H(\omega)|^2]=\frac{2u_0^2}{\omega^2T}\{2[1-|\Theta_1(\omega)|^2]+ \\ +[\Theta_1^2(\omega)-\Theta_1(2\omega)]e^{i\omega\tau_0}+[\Theta_1^2(-\omega)-\Theta_1(-2\omega)]e^{-i\omega\tau_0}\}, \quad (10.91)$$

and the discrete part

$$F_{\text{д}}(\omega)=\frac{4\pi u_0^2}{\omega^2T^2}[2|\Theta_1(\omega)|^2-\Theta_1^2(\omega)e^{i\omega\tau_0}- \\ -\Theta_0^2(-\omega)e^{-i\omega\tau_0}]\sum_{r=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi r}{T}\right). \quad (10.91')$$

If the characteristic function is even, then from (10.91) and (10.91') it follows that

$$F_{\text{н}}(\omega)=\frac{4u_0^2}{\omega^2T}\{1-|\Theta_1(\omega)|^2+[\Theta_1^2(\omega)-\Theta_1(2\omega)]\cos\omega\tau_0\}, \quad (10.92)$$

$$F_{\text{д}}(\omega)=\frac{8\pi u_0^2}{\omega^2T^2}\{|\Theta_1(\omega)|^2-\Theta_1^2(\omega)\cos\omega\tau_0\}\sum_{r=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi r}{T}\right). \quad (10.92')$$

Comparing (10.92) with (10.79), we see that the continuous spectrum with symmetrical bilateral width modulation differs from an analogous spectrum with unilateral modulation by the supplementary item $\frac{4u_0^2}{\omega^2T}[\Theta_1^2(\omega)-\Theta_1(2\omega)]\cos\omega\tau_0$

It is not difficult to write out the power spectra for the distribution of modulating white noise examined above

$$F_1(\omega)=\frac{4u_0^2}{\omega^2T}\left\{(1+e^{-\sigma^2\omega^2}\cos\omega\tau_0)(1-e^{-\sigma^2\omega^2})+ \\ +\frac{4\pi}{T}e^{-\sigma^2\omega^2}\sin^2\frac{\omega\tau_0}{2}\sum_{r=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi r}{T}\right)\right\} \quad (10.93)$$

(normal distribution),

$$F_2(\omega)=\frac{4u_0^2}{\omega^2 T}\left\{1-\frac{\sin^2\alpha\omega T}{\alpha^2\omega^2 T^2}+\left(\frac{\sin^2\alpha\omega T}{\alpha^2\omega^2 T^2}-\frac{\sin 2\alpha\omega T}{2\alpha\omega T}\right)\cos\omega\tau_0+\right.$$
$$\left.+\frac{4\pi}{T}\frac{\sin^2\alpha\omega T}{\alpha^2\omega^2 T^2}\sin^2\frac{\omega\tau_0}{2}\sum_{r=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi r}{T}\right)\right\} \tag{10.94}$$

(uniform distribution),

$$F_3(\omega)=\frac{4u_0^2}{\omega^2 T}\left\{1-J_0^2(\alpha\omega T)+\left[J_0^2(\alpha\omega T)-J_0(2\alpha\omega T)\right]\cos\omega\tau_0+\right.$$
$$\left.+\frac{4\pi}{T}J_0^2(\alpha\omega T)\sin^2\frac{\omega\tau_0}{2}\sum_{r=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi r}{T}\right)\right\} \tag{10.95}$$

(distribution of sinusoid with random phase).

In order to illustrate fully the difference of spectra with bilateral symmetrical modulation from those with unilateral, let us for instance examine the case of a uniform distribution of modulating noise and compare (10.94) with (10.85). The continuous part of spectrum $F_2(\omega)$ is shown in Figure 89. The difference from the corresponding spectrum for unilateral modulation (curve 2 in Fig. 87) is determined by the item $\frac{4u_0^2}{\omega^2 T}\left(\frac{\sin^2\alpha\omega T}{\alpha^2\omega^2 T^2}-\frac{\sin 2\alpha\omega T}{2\alpha\omega T}\right)\cos\omega\tau_0.$ The spectrum density, when ω = 0, is four times greater than with unilateral modulation. At low frequencies the supplementary item is equal approximately to $\frac{8u_0^2(\alpha T)^2}{3T}\cos\omega\tau_0.$ Therefore spectrum (10.94) differs little at these frequencies from (10.85). But at higher frequencies the difference between these spectra becomes considerable. Figure 90 shows the aperiodic part of the correlation function corresponding to $F_2(\omega)$. On account of the indicated supplementary item, the aperiodic part of the correlation function acquires two additional maxima at $\tau = \pm\tau_0$ (compare with curve 2 in Fig. 88).

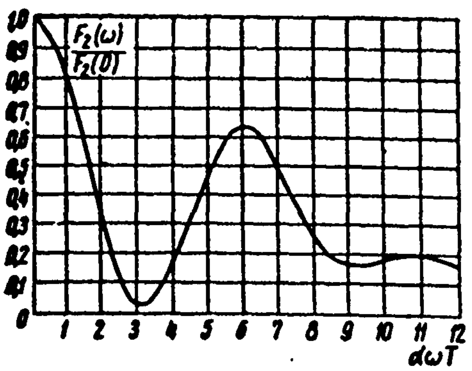


Fig. 89. Continuous portion of the power spectrum of a rectangular pulse sequence with bilateral width modulation.

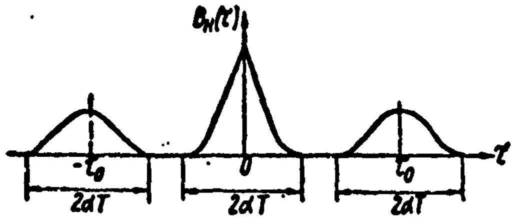


Fig. 90. Aperiodic portion of the correlation function of a rectangular pulse sequence with bilateral width modulation.

In contrast with unilateral modulation, the continuous portions of spectra and the corresponding aperiodic portions of correlation functions in this case depend upon the width, τ_0, of the unmodulated pulse.

Let us note that the discrete parts of power spectra (10.93), (10.94) and (10.95) do not differ from the corresponding discrete parts of power spectra (10.58'), (10.60) and (10.61) of sequences of rectangular pulses with constant amplitude and width emerging at random instants in time.

The modulation of rectangular pulses in position is sometimes regarded as a special case of bilateral width modulation, when the trailing and leading edges are shifted in one direction by the same amount ($\xi_n^{(1)} - \xi_n^{(2)} = \tau_0$, , where the index 1 refers to the leading edge and the index 2 to the trailing edge of the pulse). On the basis of such a point of view of modulation in position, it could be possible to repeat anew the conclusions derived from the results cited in Section 4.

8. Evaluation of the Noiseproof Qualities of Pulse Signaling Systems

Let us employ the power spectra obtained above for an evaluation of the resistance of pulse signaling systems with pulse-amplitude modulation (PAM) and pulse-width modulation (PWM) to fluctuation noise. It is known that demodulation (i.e., restoration of the modulating signal at the receiving end) is effected for other forms of pulse modulation by means of transforming the received pulses either into PAM or PWM. After this the modulating signal is segregated by the a-f filter.

The noiseproof property of a signaling system is characterized by the ratio $\frac{c}{\pi}$, by which is understood the square root of the ratio of the power of the useful signal

P_c, in the absence of interference in the pass band of the filter, to the power of the noise P_π in the same band, the latter being computed with the presence of a signal, but in the absence of useful modulation (cf. Sect. 8, Ch. VII, and also [6]).

The process at the filter input in pulse signaling systems constitutes a pulse random process. If $F_c(\omega)$ and $F_d(\omega)$ are the continuous and the discrete (except for the direct component) parts of the power spectrum of this process, and $C(\omega)$ is the frequency characteristic of the a-f filter, then it is not difficult to determine the magnitude of P_c and P_π from the formulas

$$P_c = \frac{1}{2\pi}\int_0^\infty F_д(\omega)\, C^2(\omega)\, d\omega, \tag{10.96}$$

$$P_п = \frac{1}{2\pi}\int_0^\infty F_н(\omega)\, C^2(\omega)\, d\omega. \tag{10.97}$$

For the rectangular frequency characteristics of an a-f filter with cutoff frequency Δ, these formulas are simplified and the ratio $\frac{c}{\pi}$ may be represented in the form

$$\left(\frac{c}{п}\right)^2 = \frac{\int_0^\Delta F_д(\omega)\, d\omega}{\int_0^\Delta F_н(\omega)\, d\omega}. \tag{10.98}$$

If T is the period of pulse recurrence, and τ_0 is the width of an unmodulated pulse, then, as is well known,

$$\Delta < \frac{\pi}{T},\ \Delta \ll \frac{2\pi}{\tau_0}. \tag{10.99}$$

Under these conditions in the a-f filter band, the continuous power spectrum $F_H(\omega)$ of a pulse random process does not take into account the mutual dependence of the pulses, i.e., does not differ from the spectrum of a sequence of pulses modulated by white noise.

For a PAM signalling system, employing (10.33) and taking account of (10.99), we find

$$P_н = \frac{1}{2\pi}\int_0^\Delta F_н(\omega)\, d\omega = \frac{\sigma^2}{\pi T}\int_0^\Delta |g(\omega)|^2 d\omega \approx \frac{\sigma^2}{\pi T}|g(0)|^2 \cdot \Delta.$$

The power of the signal in the filter transmission band, with modulation by a

sinusoidal signal to a depth m, is equal to

$$P_c = \frac{1}{2\pi}\int_0^{\Delta} F_x(\omega)\,d\omega = \frac{m^2a^2}{T^2}\,|g(0)|^2.$$

The ratio of signal to interference in PAM is, in accordance with (10.98) equal to

$$\frac{c}{n} = \frac{ma}{\sigma}\sqrt{\frac{\mu}{2}}, \tag{10.100}$$

where

$$\mu = \frac{2\pi}{\Delta T} > 2. \tag{10.101}$$

Formula (10.101) is the basic formula for calculating the interference-killing feature of systems with PAM in which the modulating signal is segregated by an a-f filter, and also of systems with other forms of pulse modulation in which transformation into PAM takes place.

Let us now examine the interference-killing features of a system with unilateral PWM. If αT is the maximum parasitic deviation of the leading edge due to fluctuation noise, and Δ is the filter band, then in effect $\alpha T \Delta \ll 1$. In actual fact, with a highest reproducible audio frequency of $\Delta = 2\pi \cdot 10$ kc and a maximum parasitic leading-edge deviation of $\alpha T = 0.5 \cdot 10^{-6}$ sec, the product $\alpha T \Delta = \pi \cdot 10^{-2} \approx 0.031$. Therefore, as is seen from Figure 87, in the audio-frequency range the continuous power spectrum with the unilateral width modulation of rectangular pulses is virtually uniform. Therefore

$$P_n = \frac{1}{2\pi}\int_0^{\Delta} F_n(\omega)\,d\omega \approx \frac{1}{2\pi}F_n(0)\,\Delta,$$

or, taking account of (10.87),

$$P_n = \beta\frac{u_0^2 a^2 T \Delta}{2\pi}, \tag{10.102}$$

where β is a constant factor which depends on the form of distribution of the modulating noise (for normal distribution $\beta = \frac{1}{9}$, and for uniform distribution $\beta = \frac{2}{3}$).

The signal power is

$$P_c = \frac{u_0^2}{2}\left(\frac{\tau_m}{T}\right)^2. \tag{10.103}$$

where τ_m is the maximum useful time shift of the pulse leading edge in modulation by a signal.

Substituting (10.102) and (10.103) into (10.98), we obtain

$$\frac{c}{n}=\frac{\tau_m}{aT}\sqrt{\frac{\pi}{\beta T\Delta}}=\frac{\tau_m}{aT}\sqrt{\frac{\mu}{2\beta}}, \qquad (10.104)$$

where μ is determined according to formula (10.101).

Formula (10.104) is also valid for bilateral width modulation, if only the factor 0.5 is introduced, since the spectrum density with $\omega = 0$, as has been indicated in Section 7, is four times greater for bilateral modulation as for unilateral.

Formula (10.104) is the basic formula for calculating the interference-killing features of systems with PWM, and also of systems with other forms of pulse modulation in which transformation into PWM takes place.

The maximum parasitic deviation of the leading edge (or the mean-square amount of noise) is determined on the basis of the characteristics of the communication channel in accordance with well-known formulas [5].

9. Continuous-Emission Systems

Systems of continuous emission with amplitude and angle (frequency or phase) modulation are widely used (along with pulse systems) for the transmission of messages. In these systems the processes at the output of the modulators represent high-frequency oscillation (the carrier), modulated in amplitude and phase (frequency) by a random process. This random process contains both useful information (speech, music, television, etc.) and interference (fluctuation noise); the useful part of the process can, as has already been indicated above, be determined or random.

To simplify the exposition we shall restrict ourselves to modulating processes which are normal stationary, random processes. The method set forth is, with certain computational complications, expandable for the case when the indicated processes also contain determined parts [9].

In the most general form the modulated carrier may be written analytically in the following manner (mixed modulation):

$$\xi(t) = E(t)\cos[\omega_0 t + \psi(t)], \tag{10.105}$$

where E(t) and ψ(t) are correlated random processes. Let $\xi_a(t)$ and $\xi_\phi(t)$ be normal, stationary random processes which modulate the amplitude and phase of the high-frequency carrier oscillation.

These processes are slowly-changing ones, and their power spectra are located in the low-frequency range; the highest frequency in the modulating-function spectrum which is worth taking into account is much smaller than the carrier frequency ω_0.

For the broken linear characteristic of an amplitude modulator, the link between the random function E(t) and ξ_a(t) is given by the relationship

$$\begin{aligned} E(t) &= A_0 + k\xi_a(t), \quad k\xi_a(t) \geqslant -A_0, \\ E(t) &= 0 \qquad\qquad\quad k\xi_a(t) \leqslant -A_0, \end{aligned} \tag{10.106}$$

where A_0 is the amplitude of the carrier and $m = k\left|\frac{\xi_{a\,max}}{A_0}\right|$ is the modulation factor. Since the instantaneous values of a normal random process may, on the absolute scale, be of any order of magnitude whatsoever, therefore in principle overmodulation ($m > 1$) will always be in effect. However, if the dispersion $\xi_a(t)$ is small in comparison to $\frac{A_0}{k}$, then the overmodulation may in practice be neglected. The relative time ν_T of the overmodulation is easy to evaluate from the equality

$$\nu_T = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{-\frac{A_0}{k}} e^{-\frac{x^2}{2\sigma^2}} dx = F\left(-\frac{A_0}{k\sigma}\right) = 1 - F\left(\frac{A_0}{k\sigma}\right).$$

With $\sigma \ll \frac{A_0}{k}$, employing the asymptotic expansion of the Laplace function (cf. 2.20), we obtain

$$\nu_T \sim \frac{k\sigma}{\sqrt{2\pi}A_0} e^{-\frac{A_0^2}{2k^2\sigma^2}}. \tag{10.107}$$

Conversely, if $\sigma \gg \frac{A_0}{k}$ (very strong overmodulation), then, expanding the Laplace function into an exponential series, (see 2.19), we obtain

$$\nu_T \approx \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \frac{A_0}{k\sigma}. \tag{10.107'}$$

Let the characteristics of the angle modulators be linear. Then the link between the random functions $\psi(t)$ and $\xi_\phi(t)$ is very simple. For phase modulation

$$\varphi_{\Phi M}(t)=k_1\xi_\Phi(t), \tag{10.108}$$

and for frequency modulation

$$\varphi_{ЧM}(t)=k_2\int_0^t\xi_\Phi(t)\,dt, \tag{10.109}$$

where k_1 and k_2 are respectively the characteristic slopes of the phase and frequency modulators.

It is of practical interest to find the general expression for the power spectrum of a carrier modulated in amplitude and phase by normal random processes. In solving this problem there will first be determined the correlation function, and thereupon by means of an inverse Fourier transformation - the desired power spectrum.

The subsequent computations are greatly simplified, if (10.105) is represented in the form of

$$\xi(t)=\mathrm{Re}\,\{E(t)\,e^{i\omega_0 t}e^{i\psi(t)}\}. \tag{10.110}$$

In addition, taking into account that the link between E(t) and $\xi_a(t)$ is provided by a broken linear characteristic, and employing the representation of this characteristic by a contour integral (cf. p. 284), it is possible to write E(t) in the form of

$$E(t)=\frac{1}{2\pi}\int_c e^{iu\,[A_0+k\xi_a(t)]}\frac{du}{(iu)^2}, \tag{10.111}$$

where c is the contour shown in Figure 54.

The correlation function of the random process $\xi(t)$ (cf. p 76) is equal to

$$\begin{gathered}B(\tau)=m_1\{\xi(t)\xi(t+\tau)\}=\\=\frac{1}{2}\mathrm{Re}\,e^{-i\omega_0\tau}m_1\{E(t)E(t+\tau)e^{i\psi(t)}e^{-i\psi(t+\tau)}\}.\end{gathered}$$

Designating $\xi_1=\frac{\xi_a(t)}{A_0}$, $\xi_2=\frac{\xi_a(t+\tau)}{A_0}$, $\psi_1=\psi(t)$, $\psi_2=\psi(t+\tau)$ and employing (10.111), we obtain

$$B(\tau)=\frac{A_0^2}{2}\mathrm{Re}\left[e^{-i\omega_0\tau}\frac{1}{4\pi^2}\int_c\int_{c^*}m_1\{e^{ik\xi_1u_1+ik\xi_2u_2+i\psi_1-i\psi_2}\}\frac{e^{i(u_1+u_2)}}{u_1^2u_2^2}du_1du_2\right]. \tag{10.112}$$

We introduce a four-dimensional characteristic function for the four correlated, normally distributed random variables $k\xi_1$, $k\xi_2$, ψ_1, ψ_2 (the mean values of which are assumed equal to zero)

$$\Theta_4(u_1, u_2, u_3, u_4) = e^{-\frac{1}{2}\sum_{k=1}^{4}\sum_{n=1}^{4} m_{kn} u_k u_n}, \tag{10.113}$$

where m_{kn} is the mean value of the product of the k-th and n-th random variables.

Each of these mean values has the following sense:

$$m_{11} = m_{22} = k^2 m_1 \left\{ \frac{\xi_a^2(t)}{A_0^2} \right\} = \left(\frac{k}{A_0}\right)^2 \sigma_a^2, \tag{10.114}$$

$$m_{12} = m_{21} = \frac{k^2}{A_0^2} m_1 \{\xi_a(t)\xi_a(t+\tau)\} =$$

$$= \left(\frac{k}{A_0}\right)^2 B_a(\tau) = \left(\frac{k}{A_0}\sigma_a\right)^2 R_a(\tau), \tag{10.115}$$

where σ_a^2 and $R_a(\tau)$ are the dispersion and the correlation coefficient of the modulating process $\xi_a(t)$;

$$m_{33} = m_{44} = m_1\{\psi^2(t)\} = \sigma_\varphi^2, \tag{10.116}$$

$$m_{34} = m_{43} = m_1\{\psi(t)\psi(t+\tau)\} =$$

$$= B_\varphi(\tau) = \sigma_\varphi^2 R_\varphi(\tau), \tag{10.117}$$

where σ_φ^2 and $R_\varphi(\tau)$ are the dispersion and the correlation coefficient of the random process $\psi(t)$,

$$m_{13} = m_{31} = km_1\{\xi_a(t)\psi(t)\} = \frac{k}{A_0}B_{a\varphi}(0), \tag{10.118}$$

$$m_{14} = m_{41} = km_1\{\xi_a(t)\psi(t+\tau)\} = \frac{k}{A_0}B_{a\varphi}(\tau), \tag{10.119}$$

$$m_{23} = m_{32} = km_1\{\xi_a(t+\tau)\psi(t)\} = \frac{k}{A_0}B_{\varphi a}(\tau), \tag{10.120}$$

$$m_{24} = m_{42} = km_1\{\xi_a(t+\tau)\psi(t+\tau)\} = \frac{k}{A_0}B_{\varphi a}(0), \tag{10.121}$$

where $B_{a\varphi}(\tau)$ and $B_{\varphi a}(\tau)$ are mutual correlation functions of the random process $\xi_a(t)$ and $\psi(t)$.

It is not difficult to note that

$$m_1\{e^{ik\xi_1 u_1 + ik\xi_2 u_2 + i\psi_1 - i\psi_2}\} = \Theta_4(u_1, u_2, 1, -1).$$

Employing (10.113) - (10.121), we obtain

$$m_1\left\{e^{ik\xi_1 u_1+ik\xi_2 u_2+i\psi_1-i\psi_2}\right\}=e^{-\frac{k^2\sigma_a^2}{2A_0^2}(u_1^2+2R_a u_1 u_2+u_2^2)}\times$$
$$\times e^{-\sigma_\phi^2(1-R_\phi)}\,e^{-k\{[B_{a\phi}(0)-B_{a\phi}(\tau)]u_1-[B_{\phi a}(0)-B_{\phi a}(\tau)]u_2\}}. \qquad (10.122)$$

In the right-hand part of formula (10.122) the first factor represents the two-dimensional characteristic function of the process $\frac{k}{A_0}\xi_a(t)$, the second one represents the characteristic function (with a value of the argument equal to unity) of the difference $\psi(t+\tau)-\psi(t)$ *, and the last factor takes into account the correlational link between the random processes $\xi_a(t)$ and $\psi(t)$ **.

Substituting (10.122) into (10.112), we find the expression of the correlation function of a carrier modulated in amplitude and phase by mutually correlated stationary normal random processes

$$B(\tau)=\frac{A_0^2}{2}e^{-\sigma_\phi^2(1-R_\phi)}\times$$
$$\times\operatorname{Re}\left[\frac{1}{4\pi^2}e^{-i\omega_0\tau}\int_c\int_{c^*}e^{-\frac{1}{2}\left(\frac{k\sigma_a}{A_0}\right)^2(u_1^2+2R_a u_1 u_2+u_2^2)}\times\right. \qquad (10.123)$$
$$\left.\times e^{-\frac{k}{A_0}\{[B_{a\phi}(0)-B_{a\phi}(\tau)]u_1-[B_{\phi a}(0)-B_{\phi a}(\tau)]u_2\}}\frac{e^{i(u_1+u_2)}}{u_1^2u_2^2}du_1du_2\right].$$

If the modulating processes $\xi_a(t)$ and $\psi(t)$ are noncoherent, then $B_{a\phi}=B_{\phi a}\equiv 0$, and expression (10.123) is considerably simplified and may be represented as the product

$$B(\tau)=B_A(\tau)B_\phi(\tau), \qquad (10.124)$$

where

$$B_A(\tau)=\frac{A_0^2}{2}\operatorname{Re}\left[\frac{1}{4\pi^2}e^{-i\omega_0\tau}\int_c\int_{c^*}e^{-\frac{1}{2}\left(\frac{k\sigma_a}{A_0}\right)^2(u_1^2+2R_a u_1 u_2+u_2^2)}\times\right.$$
$$\left.\times e^{i(u_1+u_2)}\frac{du_1du_2}{u_1^2u_2^2}\right]. \qquad (10.125)$$

---

* Cf. (5.62) when $\tau=0$.

** It is important to take this correlational link into account, for instance, in problems connected with the investigation of noise in magnetron generators.

is the correlation function of the carrier modulated only in amplitude, and

$$B_{\varphi}(\tau)=e^{-\sigma_{\varphi}^{2}(1-R_{\varphi})} \tag{10.126}$$

is the correlation function of the carrier modulated only in phase (or frequency).

10. Amplitude Modulation

The correlation function of an amplitude modulated carrier by a normal stationary random process is, in the case of a broken linear characteristic of the amplitude modulator, represented by formula (10.125). To compute the double integral in the right part of (10.125) it is sufficient to employ the expansion (7.38), after which the integration variables divide and the expression for $B_A(\tau)$ takes the following form

$$B_{A}(\tau)=\frac{A_0^2}{2}\cos\omega_0\tau\sum_{r=0}^{\infty}\frac{\left[\frac{k^2\sigma_a^2}{A_0^2}R_a(\tau)\right]^r}{r!}h_r^2, \tag{10.127}$$

where

$$h_r=\frac{i^r}{2\pi}\int_C u^{r-2}e^{iu-\frac{1}{2}\left(\frac{k\sigma_a}{A_0}\right)^2u^2}du. \tag{10.128}$$

Making use of the procedure employed on p. 135 , we find

$$h_r=\left(\frac{A_0}{k\sigma_a}\right)^{r-1}\varphi^{(r-2)}\left(\frac{A_0}{k\sigma_a}\right), \tag{10.128'}$$

where $\varphi(x)=\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}$.

Employing formula (8) of Appendix VI, the coefficient h_r may be expressed in terms of degenerate hypergeometric functions:

$$h_r=\frac{1}{2}\left(\frac{k\sigma_a}{A_0\sqrt{2}}\right)^{-(r-1)}\left\{\frac{1}{\Gamma\left(\frac{3-r}{2}\right)}{}_1F_1\left(\frac{r-1}{2},\frac{1}{2},-\frac{A_0^2}{2k^2\sigma_a^2}\right)+\right.$$

$$\left.+\frac{\sqrt{2}A_0}{k\sigma_a\Gamma\left(\frac{2-r}{2}\right)}{}_1F_1\left(\frac{r}{2},\frac{3}{2},-\frac{A_0^2}{2k^2\sigma_a^2}\right)\right\}=$$

$$=\frac{1}{2}\left(\frac{k\sigma_a}{A_0\sqrt{2}}\right)^{-(r-1)}\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{A_0}{k\sigma_a}\right)^m}{m!\,\Gamma\left(1-\frac{r+m-1}{2}\right)}. \tag{10.128''}$$

Substituting (10.128) into (10.127), we find

$$B_A(\tau)=\frac{k^2\sigma_a^2}{2}\cos\omega_0\tau\sum_{r=0}^{\infty}\frac{1}{r!}\left[\varphi^{(r-2)}\left(\frac{A_0}{k\sigma_a}\right)\right]^2 R_a^r(\tau). \tag{10.129}$$

Here $\varphi^{(-1)}(x)=F(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{x^2}{2}}dx$ — is a Laplace function (cf. Appendix II), and

$$\varphi^{(-2)}(x)=\int_{-\infty}^{x}F(x)\,dx.$$

Let us first examine the case when $\sigma_a \ll \frac{A_0}{k}$, in which the **overmodulation** may in practice be neglected (cf. (10.107)). In this case

$$\varphi^{(-2)}\left(\frac{A_0}{k\sigma_a}\right)\sim\frac{A_0}{k\sigma_a},\quad \varphi^{(-1)}\left(\frac{A_0}{k\sigma_a}\right)\sim 1,$$

and the rest of the items in (10.129) may be neglected. Then

$$B_A(\tau)=\frac{k^2\sigma_a^2}{2}\left[\frac{A_0^2}{k^2\sigma_a^2}+R_a(\tau)\right]\cos\omega_0\tau=$$
$$=\frac{A_0^2}{2}\cos\omega_0\tau+\frac{k^2}{2}B_a(\tau)\cos\omega_0\tau.$$

The power spectrum $F_A(\tau)$, which corresponds to $B_A(\tau)$, is equal to

$$F_A(\omega)=2A_0^2\int_0^{\infty}\cos\omega_0\tau\cos\omega\tau\,d\tau+$$
$$+2k^2\int_0^{\infty}B_a(\tau)\cos\omega_0\tau\cos\omega\tau\,d\tau=\pi A_0^2\,\delta(\omega-\omega_0)+$$
$$+k^2\int_0^{\infty}B_a(\tau)\cos(\omega_0-\omega)\tau\,d\tau+$$
$$+k^2\int_0^{\infty}B_a(\tau)\cos(\omega_0+\omega)\tau\,d\tau. \tag{10.130}$$

With the assumption that the bandwidth of the audio-frequency spectrum $F_a(\omega)$, which modulates the random process, is much less than ω_0, the last term in the right part of (10.130) may be neglected and the spectrum of the amplitude-modulated carrier may be written in the form*

$$F_A(\omega)=\pi A_0^2\,\delta(\omega-\omega_0)+\left(\frac{k}{2}\right)^2F_a(\omega_0-\omega). \tag{10.131}$$

---

* Cf. footnote, p. 275.

From (10.131) it can be seen that, in the case at hand, the spectrum of the amplitude-modulated carrier is obtained by superimposing on the carrier a discrete spectrum line at $\omega = \omega_0$ of the continuous spectrum of the modulating process, this spectrum being multiplied by a constant coefficient of $\left(\frac{K}{2}\right)^2$ and shifted into the high-frequency range by the amount ω_0. This linear link between the modulated oscillation spectrum and that of the modulating process comes about from the initial assumption of negligibly small **overmodulation**.

The other extreme case, when $\sigma_a \gg \frac{A_0}{K}$, corresponds to very strong **overmodulation** (the relative time of **overmodulation** is close to $\frac{1}{2}$, cf. 10.107'). In this case in the summation along m of formula (10.128') (sic) it is possible to restrict oneself to the first terms and then from (10.127) we find

$$B_A(\tau) = \frac{A_0^2}{2}\cos\omega_0\tau\,\frac{k^2\sigma_a^2}{A_0^2}\sum_{r=0}^{\infty}\frac{2^{r-2}R_a^r}{r!\Gamma^2\left(1-\frac{r-1}{2}\right)}.$$

Since

$$\Gamma\left(1-\frac{r-1}{2}\right) = \begin{cases} \frac{\sqrt{\pi}}{2}, & r=0, \\ 1, & r=1, \\ \infty, & r=2n+1, \\ (-1)^{n-1}\frac{2^{n-1}\sqrt{\pi}}{(2n-3)!!}, & r=2n \ (n=1,2,\ldots), \end{cases}$$

then

$$B_A(\tau) = \frac{k^2\sigma_a^2}{2}\cos\omega_0\tau\left(\frac{1}{2\pi}+\frac{R_a}{4}+\frac{R_a^2}{4\pi}+\frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{[(2n-3)!!]^2}{(2n)!}R_a^{2n}\right). \qquad (10.132)$$

The expression (10.132) differs from (7.31) only by the factor $\cos\omega_0\tau$ and therefore, taking (7.32) into account, it may be rewritten in convoluted form

$$B_A(\tau) = \frac{k^2\sigma_a^2}{4\pi}\left\{\left[\frac{\pi}{2}+\arcsin R_a(\tau)\right]R_a(\tau)+\sqrt{1-R_a^2(\tau)}\right\}\cos\omega_0\tau. \qquad (10.133)$$

From a comparison of (10.133) with (7.32), it can be seen that with strong overmodulation the spectrum of an amplitude-modulated carrier coincides with the spectrum of the modulating random process which has passed through a linear detector, this spectrum being multiplied by a constant coefficient of $\frac{K^2}{2}$ and shifted into the high-frequency range by the amount ω_0. In distinction from the case of negligibly small **overmodulation**, in the present case the modulation is a nonlinear transformation, as

a result of which in the spectrum of the modulated carrier there appear supplementary harmonics from the beats of the carrier with the components of the modulating process.

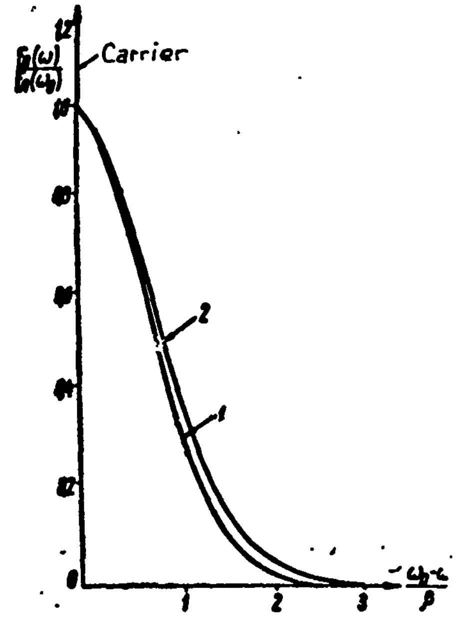


Fig. 91. Power spectrum of a carrier modulated in amplitude by a stationary normal process.

1-- negligibly small overmodulation; 2-- strong overmodulation.

Figure 91 shows the continuous spectra of a carrier modulated in amplitude by a stationary normal process, the spectrum of which has the form of a gaussian curve with a band of $\beta\sqrt{\pi}$, i.e., $F(\omega) = F_0 e^{-\frac{\omega^2}{\beta^2}}$ [9].

The first curve in Figure 1 refers to the case of negligible overmodulation and reproduces the spectrum $F(\omega)$ shifted by the amount ω_0, and the second curve refers to the case of strong overmodulation. Both curves are normalized by dividing each ordinate by $F_A(\omega_c)$. In the second case the spectrum of the modulated carrier is somewhat wider than the spectrum of the modulating process due to the products of the nonlinear transformation.

11. Frequency Modulation

The correlation function of a carrier modulated in frequency by the normal stationary process $\xi_\phi(t)$, with a linear characteristic of the frequency modulator, is with $\sigma_a = 0$ represented by formula (10.123), in which σ_ϕ and R_ϕ are the dispersion and the correlation coefficient of the random function $\psi_{FM}(t)$, linked

with $\xi_{\phi}(t)$ by the relationship (10.10?). Let us designate by $F_M(\omega)$ the power spectum of the modulating process.

Then, employing (6.11), it is not difficult to write the power spectrum $F_{\phi}(\omega)$ of the process ψ_{FM}, which is an integral of the modulating process:

$$F_{\phi}(\omega)=\frac{k_2}{\omega^2}F_M(\omega), \tag{10.134}$$

and, consequently,

$$\sigma_{\phi}^2 R_{\phi}(\tau)=B_{\phi}(\tau)=\frac{k_2}{2\pi}\int_0^{\infty}F_M(\omega)\cos\omega\tau\frac{d\omega}{\omega^2}, \tag{10.135}$$

$$\sigma_{\phi}^2=B_{\phi}(0)=\frac{k_2}{2\pi}\int_0^{\infty}F_M(\omega)\frac{d\omega}{\omega^2}. \tag{10.135'}$$

Substituting (10.135) and (10.135') into (10.126) and employing Khinchin's formula (5.44), we find the power spectrum of a carrier modulated in frequency by a normal stationary random process

$$\begin{aligned}F_{\phi}(\omega)&=4\int_0^{\infty}B_{\phi}(\tau)\cos\omega\tau d\tau=\\&=2A_0^2\int_0^{\infty}e^{-\sigma_{\phi}^2(1-R_{\phi})}\cos\omega_0\tau\cos\omega\tau d\tau=\\&=A_0^2\int_0^{\infty}e^{-\frac{k_2}{2\pi}\int_0^{\infty}(1-\cos\omega\tau)F_M\frac{d\omega}{\omega^2}}\cdot\cos(\omega-\omega_0)\tau d\tau+\\&+A_0^2\int_0^{\infty}e^{-\frac{k_2}{2\pi}\int_0^{\infty}(1-\cos\omega\tau)F_M\frac{d\omega}{\omega^2}}\cdot\cos(\omega+\omega_0)\tau d\tau.\end{aligned} \tag{10.136}$$

The second integral in the right part of (10.136) may be rejected, since in the integrand function the exponential factor changes slowly in comparison to $\cos(\omega+\omega_0)\tau$. Thus*

$$F_{\phi}(\omega)=A_0^2\int_0^{\infty}e^{-\frac{k_2}{2\pi}\int_0^{\infty}(1-\cos\omega\tau)F_M\frac{d\omega}{\omega^2}}\cos(\omega-\omega_0)\tau d\tau. \tag{10.137}$$

Let us assume that the spectrum of the modulating process is uniform in the band from zero to Δ, i.e.,

---

* Let us note that formula (10.137) yields also the spectrum of a carrier modulated in phase, if only k_2 is replaced by k_1, and the factor $\frac{1}{\omega^2}$ in the exponent under the integral sign is dropped.

$$F_{M}(\omega)=\begin{cases}\frac{2\pi}{k_2}\frac{\sigma_{FM}^2}{\Delta}, & 0<\omega<\Delta\\ 0, & \omega>\Delta,\ (\Delta\ll\omega_0),\end{cases} \tag{10.138}$$

where σ^2_{FM} is the mean-square deviation of the frequency from the carrier (the effective frequency deviation). It is customary to call the ratio of σ_{FM} to the maximum modulating frequency Δ the effective frequency-modulation index $m = \frac{\sigma_{FM}}{\Delta}$. Employing (10.138), we find from (10.137) after replacing the integration variable in the exponent

$$F_{\bullet}(\omega)=A_0^2\int_0^\infty e^{-m^2\Delta\tau\int_0^{\frac{\Delta\tau}{2}}\left(\frac{\sin y}{y}\right)^2 dy}\cos(\omega-\omega_0)\tau d\tau. \tag{10.139}$$

The integral in the exponent is expressed in terms of the integral since

$$\int_0^{\frac{\Delta\tau}{2}}\left(\frac{\sin y}{y}\right)^2 dy=\mathrm{Si}(\tau\Delta)-\frac{\sin^2\frac{\tau\Delta}{2}}{\frac{\tau\Delta}{2}}.$$

Effecting the substitution $\tau\Delta = x$, we obtain finally

$$F_{\bullet}(\omega)=\frac{A_0^2}{\Delta}\int_0^\infty e^{-m^2x\left[\mathrm{Si}(x)-\frac{\sin^2\frac{x}{2}}{\frac{x}{2}}\right]}\cos\left(\frac{\omega-\omega_0}{\Delta}x\right)dx. \tag{10.140}$$

Let us examine the two limiting conditions $m \gg 1$ and $m \ll 1$. The first condition is equivalent to the assumption that the width of the spectrum of the modulating process is narrow in comparison to the effective frequency deviation. Such conditions are realized, for instance, in radio broadcasting on ultrashort wavelengths and in television sound.

In this case, resolving the function $\sin(x)-\frac{\sin^2\frac{x}{2}}{\frac{x}{2}}=\frac{x}{2}-\frac{x^3}{72}+\cdots$, it is sufficient to restrict one's self to the first term. Then (compare p. 228)

$$F_{\bullet}(\omega)=\frac{A_0^2}{\Delta}\int_0^\infty e^{-\frac{m^2x^2}{2}}\cos\left(\frac{\omega-\omega_0}{\Delta}x\right)dx=\frac{A_0^2}{2\Delta}\cdot\frac{\sqrt{2\pi}}{m}e^{-\frac{(\omega-\omega_0)^2}{2\Delta^2m^2}},$$

or

$$F_{\bullet}(\omega)=\frac{A_0^2}{2}\cdot\frac{\sqrt{2\pi}}{\sigma_{FM}}e^{-\frac{(\omega-\omega_0)^2}{2\sigma^2_{FM}}},\ m\gg 1. \tag{10.141}$$

Thus, in the frequency modulation of a carrier by a stationary normal random process whose spectrum is narrow in comparison with the effective frequency deviation, the power spectrum of the modulated carrier has the shape of a gaussian curve with the peak at point $\omega = \omega_0$ (Fig. 92). The bandwidth of this spectrum is equal to (cf. p. 229) $\Delta_{чM}=\sqrt{2\pi}\sigma_{чM}=\sqrt{2\pi}m\Delta$.

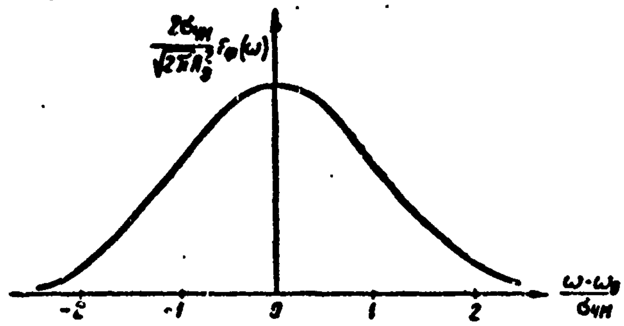


Fig. 92. Power spectrum of a carrier modulated in frequency by a stationary normal process.

In the case of narrow-band frequency modulation $m \ll 1$, a good approximation may be obtained if, in place of function sin(x)- $\frac{\sin^2\frac{x}{2}}{\frac{x}{2}}$ in (10.140) there is taken its asymptotic value when $x \longrightarrow \infty$, equal to $\frac{\pi}{2}$. Then (compare 5.57)

$$F_\Phi(\omega)=\frac{A_0^2}{\Delta}\int_0^\infty e^{-\frac{\pi}{2}m^2x}\cos\left(\frac{\omega-\omega_0}{\Delta}x\right)dx$$

or

$$\frac{A_0^2}{\Delta}\cdot\frac{\pi m^2}{2}\cdot\frac{1}{\frac{\pi^2 m^4}{4}+\frac{(\omega-\omega_0)^2}{\Delta^2}}$$

$$F_\Phi(\omega)=\frac{\pi A_0^2}{2}\frac{m\sigma_{чM}}{\left(\frac{\pi m\sigma_{чM}}{2}\right)^2+(\omega-\omega_0)^2},\quad m\ll 1. \qquad (10.142)$$

Thus, in the frequency modulation of a carrier by a stationary normal process which is wide in comparison to the effective frequency deviation, the power spectrum of the modulated carrier has the same form as the spectrum of white noise at the output of a linear RC circuit (compare p. 230). The bandwidth of this spectrum is equal to $\Delta_{чM}= \frac{\pi}{2} m \sigma_{чM} = \frac{\pi}{2} m^2\Delta$. In distinction from the preceding case, in which the ratio $\frac{\Delta_{чM}}{\Delta}$ of the spectrum band of the modulated carrier to the band of the modulating process depended linearly on the modulation index m, in the present case this relationship becomes quadratic.

Literature

1. G. S. Gorelik. On Several Magnetic Spectra of Transformation. Izv. AN SSSR, ser. fiz (USSR Academy of Sciences News, Physics Series), V. XIV, No. 2, 1950.

2. G. G. Mac Farlane. On the energy-spectrum of an almost periodic succession of pulses. Pros. IRE, 37, No. 10, 1949.

3. B. R. Levin. Metody teorii sluchaynykh protsessov, primenyayemyye v radiotekhnike (Methods of the Theory of Random Processes, Applicable to Radio Engineering), part II. Supplement to "Vestnik N11 MRTP" (Bulletin, Scientific Research Institute of the Ministry of Radio, Telegraph and Post), 1954.

4. R. Fortet. Spectre moyen d'une suite d'impulsions en principe periodiques et identiques, mais deplaceér et deformées aléatoirement. L'Onde électrique, No. 329-330, 1954.

5. V. I. Siforov and others. Teoriya impul'snoy radiosoyazi (Theory of Pulse Radio Communication). L K VVIA ("LK" Military Air Engineering Academy), 1951.

6. A. M. Petrovskiy. A Method of Calculating the Interference-killing Properties of Pulse Signaling Systems with Fluctuation Interference, Based on an Analysis of Spectrum Densities of the Distorted Signal. Avtomatika i Telemekhanika (Automation and Telemechanics), No. 2, 1954.

7. H. Kaufman, E. H. King. Spectral power density functions in pulse time modulation, Trans. IRE, v. IT-1, march. 1955.

8. E. R. Kretzmer. An application of auto-correlation analysis. Journ. of math. Phys., 29, No. 3, 1950.

9. D. Middleton. On the distribution of energy in noise and signal-modulated waves. I u II. Quart. Appl. Math. 9, No. 4, Jan. 1952, 10, No. 1, Apr. 1952.

10. I. S. Gonorovskiy. Radio signaly i perekhodnyye yavleniya v radiotsepyakh (Radio Signals and Transition Phenomena in Radio Circuits): Svyazizdat, 1954.

11. J. L. Stewart. The power spectrum of a carrier frequency modulated by gaussian noise. Proc. IRE, 42, No. 10, 1954.

Chapter XI

ELEMENTS OF INFORMATION THEORY

1. Probability-theory methods are mathematical research methods, adequate to a whole series of radio engineering problems; this has been illustrated by numerous examples in the preceding chapters. The application and development of these methods has played an important part in the progress of radio engineering science in recent years.

Along with the successful solving of specific problems, works have appeared in which the problem of the transmission of electrical signals is stated in the most general terms. The generalized statement, based on applied practice, of the problem of the effectiveness and reliability of electrical communications systems has become the starting point for a branch of probability theory at present undergoing vigorous development, namely information theory.

The purpose and scope of the present book do not permit giving an at all satisfactory survey of the present state of information theory. Only the basic concept will be set forth, and some general results will be formulated, in order to direct the attention of the reader to the chain of reasoning of this theory and to arouse an interest in the study of it.

2. The purpose of every communication system is to restore at the receiving end the message sent at the transmitting end*.

By message is meant any data (information) subject to transmission. Such information may be speech, music, an image, a text, a photo, codes, telecontrol commands, etc. In information theory the message is regarded as a random process, since if at the receiving end the content of the message is known for certain in advance, then there is no sense in effecting the transmission of such a message.

* Here is meant not only communication in the customary sense, but any form of transmission of useful data by means of electrical signals, as, for instance, radar, telemetry, automatic control systems, computing machines, etc.

12. R. G. Medhurst. The power spectrum of a carrier frequency modulated by gaussian noise. Proc. IRE, No. 43, No. 6, 1955.

13. J. L. Stewart. Frequency modulation noise in oscillators. Proc. IRE, 44, No. 3, 1956.

14. I. S. Gonorovskiy. Phase Fluctuation in A Tube Regenerator. DAN, 101, No. 4, 1955.

The device producing the message is called the message source (or information source). The statistical structure of the message is the mathematical characteristic of the message source.

By discrete source is meant a source whose product is a random process with discrete time, i.e., which represents a sequence of random variables (cf. Sect. 10, Ch. I, and also Sect. 1, Ch. V). An example of a discrete source is a device transmitting letters or code.

By continuous source is meant a source whose product is a continuous random process. An example of a continuous source is a device transmitting speech or music. In the general case the product of a continuous source may be a random function of several variables as, for instance, in three-dimensional television, where besides depending on time, the message depends also on three spatial coordinates. By means of quantizing a continuous message, or by replacing each realization of the message by a sequence of base pulses (cf. Sect. 1, Ch. V), a continuous source can be transformed into a discrete one.

For a message to be transmitted by means of a radio-engineering system, it must be transformed into an electrical signal.

An aggregate of devices along which signals pass is called a channel. The concept of a channel embraces all the technical devices which transform the signal prior to transmission, effect its transmission and reception, and complete the transformation of the received signal, as well as the medium used for transmitting the signal from the transmitter to the receiver.

If to a given signal at the input of a channel there always corresponds one and the same signal at the output of the channel, then such a channel is called an interference-free channel. In an interference-free channel the signals at both ends of the channel are linked by a definite functional relationship. The problem of communication in this case is reduced to the identification of the received signal with one of the possible realizations of the message.

The presence of interference (noise) brings about an irregularity in the

indicated functional relationship between the message sent and the signal received. In the same manner as a message, interference is regarded as a random process. Therefore cases are possible when various signals at the output of a channel will correspond to one and the same message, and conversely, it may be possible to identify the signal received with several possible realizations of the message.

Both the source and the channel may be characterized by their fine statistical structure, i.e., probability distribution functions may be assigned to them. For the source these will be a priori distribution functions of the message $\xi(t)$, and for a channel with interference they will be a posteriori distribution functions of the probability of obtaining signal $\eta(t)$ at the output, under the condition that message $\xi(t)$ was sent (or a posteriori distribution functions of the probability that message $\xi(t)$ was sent if signal $\eta(t)$ was received). However, for comparing various sources and channels, some numerical characteristics of these distribution functions are used in information theory.

3. Let us examine a discrete message source. We assume that the number of various elements, of which any realization of the message may be composed, is finite. The aggregate of these elements, x_i ($i = 1, 2, \ldots m$) is called the alphabet of the source. An alphabet may be an aggregate of various letters of a given language, of code symbols, or of conventional commands. Any realization of a message represents a sequence of elements, each of which may be repeated an arbitrary number of times. In the realization, the same elements will thus be distinguished by the place they occupy in the sequence. Therefore in distinction from an element in an alphabet, it is expedient to introduce a separate name - symbol - for an element in the realization of a message.

Let the various realizations of message $\xi(t)$ ($t = t_0, t_0 + 1, \ldots t_0 + n - 1$) be composed of n consecutive symbols and let P_j be the probability of the appearance of realization $\xi_j(t)$. The number of various realizations consisting of n symbols is, with the given alphabet of m symbols, obviously equal to $N = m^n$. If the appearance of each realization is treated as a separate event, then the aggregate of N real-

izations will comprise a complete set of events $\left(\sum_{j=1}^{N} p_j = 1\right)$.

The amount $-\ln P_j$ is (by definition) called the quantity of information about event ξ_j. The mean quantity of information for one realization is equal to

$$H_n = -\sum_{j=1}^{N} P_j \ln P_j, \tag{11.1}$$

i.e., coincides with the entropy of the law of distribution of the message (cf. Sect. 11, Chapter II). Here the average quantity of information for one symbol delivered by the source is equal to $\frac{H_n}{n}$. If the random process $\xi(t)$ is stationary, then there exists a limit

$$H = \lim_{n\to\infty} \frac{H_n}{n}, \tag{11.2}$$

which depends only on the nature of the source and is called the entropy of the source.

Let there be known the average number M of the symbols delivered by the source in a unit of time. Then it is possible to introduce also the entropy of the source in a unit of time, by the formula

$$H' = MH. \tag{11.3}$$

The magnitude H', being the mean quantity of information in a unit of time, may be called the speed of delivery of information by the source.

If all the symbols in the realizations are mutually independent, and $p_1, p_2, \ldots p_m$ are the a priori probabilities of appearance of the elements of the alphabet, then with a sufficiently large n the probability of appearance of any realization will be equal to

$$P_j = P = p_1^{np_1} p_2^{np_2} \ldots p_m^{np_m}.$$

and, consequently, the mean quantity of information for one realization is

$$H_n = -\ln P = -n \sum_{i=1}^{m} p_i \ln p_i. \tag{11.4}$$

The source entropy is in this case equal to

$$H_{(1)} = -\sum_{i=1}^{m} p_i \ln p_i \qquad (11.5)$$

In the general case $k \leq n$ symbols are statistically linked by a k-dimensional distribution law. Such a group of symbols is called a k-gram. Monograms ($k = 1$) are symbols themselves in case they are mutually independent. Diagrams are groups of two symbols, linked by the two-dimensional distribution law $P_{i,j}$. The source entropy for one diagram is equal to

$$H_{(2)} = -\sum_{i=1}^{m} \sum_{j=1}^{m} p_{ij} \ln p_{ij}, \qquad (11.6)$$

and the entropy for a symbol is $H^{(2)} = \frac{1}{2} H_{(2)}$. Analogously the source entropy for a k-gram is

$$H_{(k)} = -\sum_{i_1=1}^{m} \cdots \sum_{i_k=1}^{m} p_{i_1 \ldots i_k} \ln p_{i_1 \ldots i_k}, \qquad (11.7)$$

and the entropy for a symbol is $H^{(k)} = \frac{1}{k} H_{(k)}$.

Bearing in mind the inequality (2.120), it is not difficult to conclude that $H^{(k)} \leq H_{(1)}$ and analogously $H^{(k)} \leq H_{(k-1)}$, i.e., taking into account the statistical interdependence of the symbols reduces the source entropy.

4. The simplest discrete source is one which delivers sequences of symbols consisting of two possible elements. An example of this is a source of telegraphic messages, whose alphabet consists of two elements: "yes" or "no" (cf. Sect. 4, Ch. I, and also Fig. 2). If all the symbols are mutually independent, then the probability of the appearance of any one of $N = 2^n$ realizations, consisting of n symbols, will be equal to $P = p^r q^{n-r}$, where p and $q = 1 - p$ are a priori probabilities of the appearance of each of two possible elements and r is any whole number not exceeding n. However, with a sufficiently large n, $r \approx np$ and, consequently the mean quantity of information for one realization will comprise

$$H_n = -\ln P = -\ln (p^r q^{n-r}) \approx -\ln (p^{np} q^{nq}) = \\ = -n(p \ln p + q \ln q),$$

and per symbol delivered

$$H = \frac{H_n}{n} = -(p \ln p + q \ln q). \qquad (11.8)$$

The quantity H in (11.8) does not depend on n and represents the entropy of the telegraphic message source.

For the data of the example in #4, Chapter I, p = P("yes") = 5/8, q = P("no") = 3/8, the source entropy is equal to $H = -(5/8 \ln 5/8 + 3/8 \ln 3/8) = 0.67$ natural unit/symbol.

5. If all the elements of the alphabet are equiprobable, then with a sufficiently large n all N realizations of the message will also be equiprobable. In that case, as has been indicated in Sect. 11, Ch. II, the entropy H_n is at its maximum and is equal to $(H_n)_{max} = \ln N = n \ln m$. The source entropy with a given number of m elements will in this case be equal to its maximum value of $H_{max} = \ln m$.

In the general case some elements will be more probable, and others less so, and the same property will be possessed by the realizations, as a result of which $H_n \leqslant (H_n)_{max}$. With a given H_n it would have been possible to obtain the same amount of information with a smaller number of $N_1 < N$ realizations, if they had all been equiprobable. Therefore the quantity $H_{max} - H$ is called interior (or excess) information, and the ratio

$$R_u = \frac{H_{max} - H}{H_{max}} = 1 - \frac{H}{H_{max}} = 1 - \frac{H}{\ln m}. \qquad (11.9)$$

is called the coefficient of excess, or simply the excess, of the source. In the example indicated above $H_{max} = \ln 2 = 0.7$ natural unit/symbol, and the coefficient of excess is equal to $R_u = 1 - \frac{0.67}{0.7} = 0.043$.

As has been noted above, the source entropy is further diminished, and consequently, the excess is further increased, if between the symbols or groups of symbols there exist statistical relationships. Thus, for the alphabet of the English language, consisting of twenty-six letters, the maximum entropy is equal to $H_{max} = \log_2 26 = 4.7$ binary units/symbol*. If it is taken into account that the letters are not equiprobable, but remain independent (monogram), then the entropy falls to 4.14

* Cf. footnote, p. 82.

binary units/symbol. The additional consideration of the statistical structure of the language yields a corresponding reduction of the entropy to 3.56 binary units/symbol (diagram) and 3.3 binary units/symbol (triogram). If account is taken of the probability of the appearance of individual words in a complete sentence, the entropy falls to a quantity smaller than one binary unit per symbol, which corresponds to an excess of more than 78%.

6. One of the general results of information theory is the theorem of the elimination of excess in a message by means of appropriate coding. By coding is meant any mutually unique transformation of the realizations ξ_j of message $\xi(t)$ into the realization η_k of another message (or signal) $\eta(t)$, i.e., from the mathematical point of view coding is a functional transformation of the message.

It is clear that when coding with utmost brevity the more frequently encountered realizations and employing longer code combinations for realizations rarely encountered, it becomes possible for us to reduce the average number of symbols in the coded realization in comparison to the original one. Let the realization ξ_j, consisting of n successive symbols and having a probability of P_j, be coded into the realization η_j, consisting of n_j symbols. The ratio $\frac{\eta_j}{\eta}$ may be regarded as the coefficient of compression of the given realization. The mean statistical value of this ratio for all possible realizations ξ_j, consisting of n symbols, is equal to

$$\mu_n = \frac{1}{n}\sum_{j=1}^{N} n_j P_j. \tag{11.10}$$

The limit

$$\mu = \lim_{r \to \infty} \mu_n \tag{11.11}$$

is called the coefficient of compression of a message with a given code.

It has been proven (cf. [13]), that if the entropy of a message is equal to H, then the smallest possible value of the coefficient of compression is equal to $1 - R_u = \frac{H}{H_{max}} = \frac{H}{\ln m}$. Employing this theorem, it is possible to find the bottom limit to the compression of a message which may be accomplished by means of an optimum

codification. Thus, in the above-indicated example of the telegraph message, the excess is extremely small and the possible compression cannot exceed 5%, whereas with a 78% excess an English text permits a compression of 4.5 times.

The theorem just formulated and the existing methods of proving it establish the possibility of a compression limit, but give no indications as to how the optimum code should be arrived at. With the practical application of various methods of codification for compressing a message, it is possible on the basis of this theorem to evaluate the extent to which the method employed approaches the optimum.

7. We pass to the quantitative character of a channel. The connection of a source, producing the messages ξ_j ($j = 1, 2, \ldots N$) with a priori probabilities of $P(j)$, to a channel leads to the appearance at the output of the channel of a new source, producing the signals η_k ($k = 1, 2, \ldots N$) with the a posteriori probabilities $P_j(k)$. If at the input of the channel the quantity of information about the realization ξ_j was measured by the quantity $-\ln P(j)$, then at the output of the channel the quantity of information about this realization is equal to $-\ln P_k(j)$. Thus the quantity of information about the realization ξ_i has changed in the channel by the amount $-\ln P(j) + \ln P_k(j) = \ln \frac{P_k(j)}{P(j)}$. The mean quantity of information for one realization of the transmitted message, contained in the signal at the output of the channel, is equal to*

$$I_n\{\xi, \eta\} = \sum_{k=1}^{N} P(k) \sum_{j=1}^{N} P_k(j) \ln \frac{P_k(j)}{P(j)}. \tag{11.12}$$

Substituting by the rule of multiplication $P_k(j) = \frac{P(j,k)}{P(k)}$, it is possible to rewrite (11.12) in a form symmetrical with respect to ξ and η

$$I_n\{\xi, \eta\} = \sum_{k=1}^{N} \sum_{j=1}^{N} P(j, k) \ln \frac{P(j, k)}{P(j) P(k)}. \tag{11.13}$$

If the interference in the channel is so strong that the signal η at the output of the channel becomes independent of the transmitted message ξ, then $P(j, k) = P(j) P(k)$, and from (11.13) it follows that $I_n = 0$, i.e., such a channel transmits no

* The curved brackets { } indicate not a function of the random variables ξ and η, but some numerical characteristic linked with these variables.

information about the message ξ. Conversely, for a channel without interference it is always possible uniquely to determine the transmitted message on the basis of the signal received. Therefore for a channel without interference $P(j) = 1$ if $k = j$, and is equal to zero with $k \neq j$. Then from (11.12) it follows that $I_n\{\xi, \xi\} = -\sum_{j=1}^{N} P(j) \ln P(j) = H_n\{\xi\}$, i.e. in this case the quantity of information contained in signal η at the output of the channel, about the message ξ, is equal to the source entropy H_n, or to the quantity of information about ξ contained in ξ itself.

Introducing conventional entropies (Sect. 11, Chapter II), it is not difficult to find from (11.13) another expression for the mean quantity of information

$$I_n\{\xi, \eta\} = -\sum_{k=1}^{N}\sum_{j=1}^{N} P(k) P_k(j) \ln P_k(j) -$$

$$-\sum_{k=1}^{N}\sum_{j=1}^{N} P(j, k) \ln P(j) = -\sum_{k=1}^{N} P(k) H_{nk}\{\xi\} - \sum_{j=1}^{N} P(j) \ln P(j),$$

or

$$I_n\{\xi, \eta\} = H_n\{\xi\} - H_{n\eta}\{\xi\}, \qquad (11.14)$$

where

$$H_{n\eta}\{\xi\} = \sum_{k=1}^{N} P(k) H_{nk}\{\xi\} = m_1\{H_{nk}\{\xi\}\} \qquad (11.15)$$

is the average (for all possible instances of k) value of the a posteriori entropy of the message.

The quantity $H_{n\eta}\{\xi\}$ shows how great the mean indeterminacy for one realization is, in determining the transmitted message on the basis of the signal received after transmission over a channel with interference. This quantity, called uncertainty, is an important characteristic of interference in a channel. For a channel without interference $H_{n\eta}\{\xi\} = 0$, i.e., the uncertainty is equal to zero. This signifies that between the possible realizations of the signal and the message there exists a functional congruence. Therefore, for instance, coding may be regarded as transmission along a channel without interference.

If the source at the input of a channel is stationary and source resulting from

connection with the channel is also stationary, then there exist the limits

$$H\{\xi\} = \lim_{n\to\infty} \frac{1}{n} H_n\{\xi\}, \quad H_\eta\{\xi\} = \lim_{n\to\infty} \frac{1}{n} H_{n\eta}\{\xi\}.$$

Then the quantity

$$I\{\xi, \eta\} = \lim_{n\to\infty} \frac{1}{n} I_n\{\xi, \eta\} = H\{\xi\} - H_\eta\{\xi\} \tag{11.16}$$

represents the mean quantity of information obtained in the transmission over the channel of one symbol.

Introducing, in accordance with (11.3), entropy per unit of time in place of entropy per symbol, we obtain the mean increment of information per unit of time

$$I'\{\xi, \eta\} = H'\{\xi\} - H'_\eta\{\xi\}, \tag{11.17}$$

which characterizes the speed of transmission of information over a channel from a given source.

The full symmetry of (11.13) with respect to ξ and η makes it possible to represent the mean quantity of information per symbol in the form of

$$I\{\xi, \eta\} = H\{\eta\} - H_\xi\{\eta\}, \tag{11.18}$$

where $H\{\eta\}$ - is the a priori source entropy at the output of the channel,

$H_\xi\{\eta\}$ - is the average value, for all possible message realizations, of the a posteriori entropy of the signal at the output of the channel.

3. For an illustration of the introduced concept let us turn again to the example in Sect. 4, Ch. I. At the input of the channel there enter symbol sequences consisting of "yes" or "no" elements, the a priori probabilities of which are P("yes") = 5/8 and P("no") = 3/8. At the output of the channel symbol sequences are observed which also consist of two possible elements: "green" and "red", the a priori probabilities of which are equal to each other P("green") = P("red") = ½. The a posteriori probabilities of the transmission of message elements in the reception of a signal element are equal to

P"green" ("yes") = 3/4, P"green" ("no") = ¼

P"red" ("yes") = ½, P"red" ("no") = ½

The amount of unreliability per symbol is equal to

$$H_{\eta}\{\xi\} = -\frac{1}{2}\left(\frac{3}{4}\ln\frac{3}{4} + \frac{1}{4}\ln\frac{1}{4}\right) - \frac{1}{2}\left(\frac{1}{2}\ln\frac{1}{2} + \frac{1}{2}\ln\frac{1}{2}\right) =$$

$$= 0.63 \text{ natural unit/symbol}$$

The entropy of the source under examination has above been found equal to $H\{\xi\} = 0.67$ natural unit/symbol. Then on the basis of formula (11.16) we find the mean quantity of information obtained for one symbol at the output of the channel $I = 0.67 - 0.63 = 0.04$ natural unit/symbol.

9. By the carrying capacity of a channel is meant the maximum possible speed of transmission of information

$$C = I'_{max}\{\xi, \eta\}. \tag{11.19}$$

The maximum is taken on the basis of all the possible message sources which may transmit signals over the channel. This quantity must depend only on the channel properties and is the most important characteristic of the channel.

For a channel without interference the unreliability $H_{\eta}\{\xi\} = 0$, and then

$$C = H'_{max}.$$

Since the coding of a message may be regarded as its transmission over a channel without interference, it therefore follows from the theorem on message compression (cf. 6 above) that by means of appropriate coding it is possible to bring the source entropy to a maximum, equal to the carrying capacity of the channel.

10. A most important result of information theory is Shannon's Theorem on the optimum utilization of channels with interference by means of appropriate coding of the transmitted message.

The difference should immediately be emphasized between coding in a channel without interference for maximum message compression, and coding in a channel with interference with the purpose of combatting interference. In the first case the coding is directed at removal of the excess present in the message, while in a channel with interference transmission with excess symbols is a reliable means of reducing mistakes. The simplest (but not the best) method of introducing excess is multiple

repetition. In an ordinary text, due to the high excess, errors in the transmission of individual letters are easily corrected on the basis of meaning.

It could be expected that as the requirements of transmission reliability are increased, the transmission speed should approach zero. Shannon has shown that, in actual fact, the carrying capacity of a channel has a fully determined value, different from zero, no matter how low the error frequency may be. In Shannon's Theorem two quantities are employed in evaluating the interference-proof features of a channel - the probability of error, and unreliability (i.e., the mean value of the a posteriori message entropy). Therefore, as has been justly pointed out by A. Ya. Khinchin [6], it is preferable to speak of two theorems, without uniting them under a common heading.

Theorem I. Let C be the carrying capacity the channel, and $H' < C$ be the rate of production of information by the source, and let $\varepsilon > 0$ be of any desired degree of smallness. It is possible to code the output of the source $\xi(t)$ into a signal $\xi_1(t)$ in such a manner, that each realization ξ_i of the message, consisting of a sufficiently large number of n symbols, will turn into realization ξ_{1i} of the signal of n + n symbols and that, in transmitting the signal over the channel, it is possible upon its realization at the output, with a probability exceeding $1 - \varepsilon$ correctly to determine the message sent.

Theorem II*. Let C be the carrying capacity of the channel, and $H' < C$ be the rate of production of information by the channel. It is possible so to code the output of the source that the rate of information transmission I' would be as close as desired to H', i.e., to the rate of the production of this information by the source. In view of (11.17) this indicates the possibility of transmitting a message over the channel with as little unreliability as may be desired.

It should be noted again that, as also in the case of the theorem on message compression, the cited theorems indicate only the possibility of optimum coding, but

* As has been shown in [12], the second theorem may be obtained as a consequence of the first.

not the means of its practical realization. These theorems are of the nature of limit theorems (as are also Lyapunov's theorem and the law of large numbers). The following problem of practical significance is to find convenient approximations to the limit relationships, as has already been done for the central limit theorem and in error theory for the law of large numbers (cf. Chapter IV).

11. In order to emphasize the general nature of information theory, it may be pointed out that from the point of view of this theory, it is also possible to examine problems dealing with the reliability of systems consisting of a large number of elements (the simplest of which, cf. in Sect. 4, Ch. I). There is a complete analogy between the concepts of "message" and "systems", "message symbol" and "system elements", between a statistical description of an interruption in the functioning of a system element and a statistical description of a channel with interference. A system consisting only of elements in series is analogous to a message entirely lacking in excess symbols. Reservation, i.e., the introduction of excess elements in a system which functions in parallel, is the most effective method of increasing its reliability. This, however, leads to an increase of the physical volume of the equipment and of its cost, which in a certain sense is equivalent to decreasing the amount of information for one symbol. It would apparently make definite sense to formulate problems of system reliability mathematically in the same form as problems in message transmission, and to apply to this case the general theorems of information theory.

12. The concept of the quantity of information in transmission over a channel is generalized for the case of continuous sources. Referring the reader to works [1], [14], [17] for general and rigorous generalizations, we shall restrict ourselves to the simplest ones.

The probability of an event, consisting in the fact that the values of the continuous message $\xi(t)$ are located on the interval of $(x, x + dx)$, is equal to $w_{11}(x)\,dx$, and the probability of the same event, under the condition that at the output of the channel the signal $\eta(t)$ is observed, is equal to $w(x/y)dx$. The increment in the amount of information is equal to $\ln \frac{\omega(x/y)}{\omega_{11}(x)}$. The mean quantity of information

with respect to message ξ (t), contained in signal η (t), is equal to

$$I\{\xi,\eta\}=\int_{-\infty}^{\infty}w_{12}(y)\int_{-\infty}^{\infty}w(x/y)\ln\frac{w(x/y)}{w_{11}(x)}dxdy, \tag{11.20}$$

or

$$I\{\xi,\eta\}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}w_2(x,y)\ln\frac{w_2(x,y)}{w_{11}(x)w_{12}(y)}dxdy. \tag{11.21}$$

Analogously to the manner in which (11.14) was obtained from (11.13), from formula (11.21) with consideration of (2.116) we obtain

$$\begin{gathered}I\{\xi,\eta\}=-\int_{-\infty}^{\infty}w_{11}(x)\ln w_{11}(x)\,dx+\\+\int_{-\infty}^{\infty}w_{12}(y)H_{x/y}dy=H\{\xi\}-H_{\eta}\{\xi\},\end{gathered} \tag{1 }$$

where H $\{\xi\}$ is the a priori entropy of message ξ (t), and $H_{\eta}\{\xi\}$ is the mean value of the a posteriori entropy of message ξ (t), under the condition that signal η (t) was received.

The quantity $H_{\eta}(\xi)$ retains the designation of unreliability.

The full symmetry of (11.21) with respect to x and y makes it possible also to write

$$\begin{gathered}I\{\xi,\eta\}=-\int_{-\infty}^{\infty}w_{12}(y)\ln w_{12}(y)\,dy+\\+\int_{-\infty}^{\infty}w_{11}(x)H_{y/x}dx=H\{\eta\}-H_{\xi}\{\eta\}.\end{gathered} \tag{11.23}$$

For channels with a limited transmission band it is possible also to introduce the concept of rate I' of information transmission, if I is multiplied by the number of data (excerpts) M per unit of time by which are determined those signals whose spectrum occupies the limited frequency band. This number M is determined on the basis of the well-known Kotelnikov's theorem (cf. Footnote, p. 186) and is equal to M = 2F data/sec, where F is the highest frequency in the signal spectrum. Thus

$$I'\{\xi,\eta\}=2FI\{\xi,\eta\}. \tag{11.24}$$

It should, however, be kept in mind that between the discrete and continuous cases there are a number of substantial differences. Thus, for instance, functional transformations of symbol sequences can change their length, but do not change the

probability of emergence and, consequently, the entropy for one realization. The functional transformation of a continuous message changes the probability distribution function and with it the entropy of the message. If $H\{\xi\}$ is the entropy of message $\xi(t)$, then, employing (2.114) and (3.11), it is not difficult to write the entropy of signal $\eta(t)$

$$H\{\eta\}=H\{\xi\}-\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}w_n(x_1,\dots x_n)\ln|D|\,dx_1\dots dx_n,$$

where $w_n(x_1,\dots x_n)$ is the n-dimensional distribution function of $\xi(t)$, and D is the transformation jacobian of the variables x_i to the variables $y_i = f(x_1,\dots x_n)$.

Only in transformations for which $|D| = 1$ (for instance, the addition of $\xi(t)$ with an arbitrary determined function of time) does the entropy retain its value. Thus, in effecting functional transformations on continuous messages, it is possible by means of changing the distribution function ("redistribution" of the probabilities) to attain the desired change in entropy.

13. Let us assume that the transmitted message ξ and the interference acting in the channel ζ are independent random processes, and let $w_{13}(z)$ be the distribution function of the interference. Since the signal at the output of the channel is $\eta = \xi + \zeta$, therefore

$$w(y/x) = w_{13}(y - x)$$

and

$$H_\xi\{\eta\}=m_1\left\{\int_{-\infty}^{\infty}w_{11}(x)\int_{-\infty}^{\infty}w(y/x)\ln w(y/x)\,dy\,dx\right\}=$$
$$=m_1\left\{\int_{-\infty}^{\infty}w_{11}(x)\int_{-\infty}^{\infty}w_{13}(y-x)\ln w_{13}(y-x)\,dy\,dx\right\}=$$
$$=\int_{-\infty}^{\infty}w_{13}(z)\ln w_{13}(z)\,dz,$$

or

$$H_\xi\{\eta\}=H\{\zeta\}, \tag{11.25}$$

i.e., the mean value of the a posteriori entropy of the received signal η, under the condition that message ξ was sent, is equal to the entropy of the interference.

Substituting (11.25) into (11.23), we obtain

$$I\{\xi, \eta\} = H\{\eta\} - H\{\zeta\}. \tag{11.26}$$

Since $H\{\zeta\}$ does not depend on the message source, the maximum value of $I\{\xi, \eta\}$ is attained at the maximum of entropy $H\{\eta\}$ of the received signal. Therefore, taking into account (11.1?) and (11.24), it is possible in the case under examination to compute the carrying capacity of a channel, wherein the signal-and-interference band is limited by a maximum frequency F, by means of the formula

$$C = 2F[H_{max}\{\eta\} - H\{\zeta\}]. \tag{11.27}$$

Let us suppose that the interference is distributed according to the normal law with a dispersion of σ_{π}^2, and let us assume that the mean powers of the transmitted signals are limited by the quantity σ_c^2.

In view of the independence of the interference and signals, the mean powers of the signals received are limited by the quantity $\sigma_c^2 + \sigma_{\pi}^2$. As has already been noted in #11, Chapter II, a continuous random variable with a limited dispersion has maximum entropy if it is distributed normally. With respect to the problem at hand, this signifies that the rate of information transmission in the channel will be at its maximum if the signals at its output are normally distributed. Since the interference is independent of the signals and is also normal, the latter is equivalent to a requirement for a normal distribution of the transmitted signals.

According to (2.113) the entropy of a normally distributed signal with a dispersion of $\sigma_c^2 + \sigma_{\pi}^2$ is equal to

$$H_{max}\{\eta\} = \frac{1}{2}\ln[2\pi e(\sigma_c^2 + \sigma_{\pi}^2)],$$

and the entropy of the interference is

$$H\{\zeta\} = \frac{1}{2}\ln(2\pi e\sigma_{\pi}^2).$$

Therefore in accordance with (11.27) the carrying capacity of a channel with band F, in which there acts a normally-distributed interference with a mean power of σ_{π}, is equal (with a mean power of the transmitted signals of σ_c^2) to

$$C = F \ln \frac{\sigma_c^2 + \sigma_\pi^2}{\sigma_\pi^2} = F \ln\left(1 + \frac{\sigma_c^2}{\sigma_\pi^2}\right). \qquad (11.28)$$

It can be seen from (11.28), that if the interference ζ is weak ($\sigma_\pi \ll \sigma_c$), then the quantity of information about ξ, when $\eta = \xi + \zeta$ is known, is great and becomes unlimited when $\sigma_\pi \longrightarrow 0$.

Conversely, the quantity of information rapidly diminishes with an increase in the interference dispersion. It also follows from this formula that with a decrease in the signal/interference ratio $\frac{\sigma_c}{\sigma_\pi}$, it is possible to preserve the carrying capacity of the channel by widening its band F and, conversely, it is possible to narrow band F at the cost of increasing the signal/interference ratio.

Employing Shannon's Theorem (10, above) it is possible to assert further that with appropriate coding of the message it will be possible to transmit signals with as small a probability of error as desired if only the rate of transmission over the channel does not exceed its carrying capacity, which is determined by formula (11.28). To attain the maximum transmission rate the message must be coded in such a manner that the transmitted signals have a normal law of distribution.

14. The general aspect of information theory makes it possible to specify the following groups of problems, which are of importance in setting up or analyzing any communications system *.

1) Optimum coding of the transmitted message into a signal possessing minimum excess. The elimination of excess is attained by limitation, by the employment of a priori information (such as the data of the system itself or the shape and parameters of the signals employed), and also by means of the elimination of statistical interrelationships in the message (decorrelation). A whole series of decorrelation methods is known: consolidation (codes with a lag), linear prediction, dynamic codes, spectral codes (emphasis of individual frequencies), spectrum compression, etc..

---

* Cf. footnote p. 426.

The reduction of excess increases the efficiency of a system, but lowers its reliability. A second group of problems arises.

2) Optimum coding of the transmitted signal to provide the most interference-proof transmission of this signal over a channel with noise. Here belong, above all, various methods of modulation in wide-band systems (frequency, pulse, code, etc.).

3) The third group of problems is linked with the question of the practical possibility of extracting the useful information contained in the signal at the output of a channel, i.e., with the possibility of separating the transmitted signal from interference. The indicated possibility is realized by various methods of separating the signal out at the receiving end: amplitude limitation, frequency (optimum filters, comb filters) and time methods (storage, correlational reception). These are joined by methods of error detection in the received signal and of the automatic correction of these errors by means of corrective codes.

15. In connection with problems of reception, considerable interest is afforded by the following problem. It is possible, according to Shannon's Theorem, so to code a message into a signal transmitted over a channel with interference, that on the basis of the received signal it will be possible to determine the transmitted message with as small a probability of error as desired, so long as the rate of information transmission does not exceed the carrying capacity of the channel. However, in a number of cases the communication system is given, and there is no possibility of arbitrarily coding the transmitted message. Let us therefore assume that the statistical characteristics of the transmitted signal and of the interference in the channel are given. What in such a case is the optimum method of reception, and what are those minimum probabilities of error in the determination of the transmitted signal on the basis of the received one which may be attained by means of receiver improvement?

This problem was formulated and solved by V. A. Kotel'knikov [2]. We shall pause briefly on the basic idea of the cited work, restricting ourselves to the simplest case of the transmission of a discrete signal, any of whose N realizations

consists of a sequence of n symbols.

Each of the symbols of the sequence may be treated as a coordinate of a given realization of the transmitted signal in a space of n dimensions. Then to each realization there corresponds a point in the n-dimensional space, and to the aggregate of all the possible realizations of the transmitted signal there correspond N various points $a_1, a_2, \dots a_N$ of this space. At the output of the channel the realizations of the signal will also represent sequences of n symbols, to which will correspond N points of the space $x_1, x_2, \dots x_N$. The points in space corresponding to the signal at the channel output will not, however, coincide with the points corresponding to the transmitted signal, and between the points a_j and x_k there is not, due to the random character of the interference, even a functional relationship. The problem of reception lies in restoring the transmitted signal a_j on the basis of the received signal, i.e., on the basis of point x_k.

Let us break down the entire space of n dimensions into N non-overlapping regions $G_i (i = 1, 2, \dots N)$ and let us consider that if the received signal x_k has fallen into a region G_i, then the signal a_i has been transmitted (compare with Sect. 5, Chapter VIII). With some probabilities the point x_k may fall into any of N regions. In case this point corresponds to the transmitted signal a_i and does not fall into a region G_i, the receiver erroneously reproduces a signal different from a_i. Depending on the configurations of the regions G_i the probabilities of error will be greater or less. It is possible to formulate the problem of determining an optimum breakdown of the signal space into the regions $G_i (i = 1, 2, \dots N)$, for which the probability of the correct reproduction of the sent signal is the maximum one. A receiver which effects an optimum breakdown of the space, and which yields the minimum number of incorrectly reproduced signals, is called an ideal one according to Kotel'nikov. The interference-resistant quality of an ideal receiver, attainable as a limit for given statistical characteristics of signal and interference, is called the potential interference-resistant quality.

In the simplest case, when all the realizations of a signal are equiprobable,

the optimum breakdown of the space is one in which any point x of the space refers to a region G_i of that signal, to whose representational point a the given point is closest of all. This same rule for the construction of an ideal receiver is also retained for nonequiprobable realizations of a signal, but under the condition that the intensity of the interference be sufficiently low.

Literature

1. C. E. Shannon, W. Weaver. The mathematical theory of communication, 1949. (A translation [into Russian] is available in the symposium "Teoriya peredachi elektricheskih signalov pri nalichii pomekh" (Theory of the Transmission of Signals in the Presence of Interference) edited by N. A. Zheleznov, For. Lit. Pub. Hse., 1953).

2. V. A. Kotel'nikov. Teoriya potentsial'noy pomekhoustoychivosti pri fluktuatsionnykh pomekhakh (Theory of the potential interference-resistant quality with fluctuation noise). Doctoral dissertation, 1946.

3. A. A. Kharkevich. Ocherki obshchey teorii svyazi (Outlines of General Communications Theory). Gostekhizdat, 1955.

4. F. M. Vudvord (Woodward). Teoriya veroyatnostey i teoriya informatsii s primeneniyami k radiolokatsii (Probability Theory and Information Theory with Applications to Radar). Translation from English, edited by G. S. Gorelik. "Sovetskoye Radio" (Soviet Radio) Publishing House, 1955.

5. M. P. Dolukhanov. Vvedeniya v teoriyu peredachi informatsii po elektricheskim kanalam svyazi (Introduction to the Theory of the Transmission of Information along Electrical Communication Channels). [No date.]

6. A. Ya. Khinchin. On the Basic Theorem of Information Theory. "Uspekhi Matematicheskikh Nauk ("Successes of the Mathematical Sciences"), V. XI, No. 1, 1956.

7. S. Goldman. Teoriya informatsii (Information Theory) Trans. from Engl. edited by V. V. Furdtsev. For. Lit. Pub. Hse., 1957.

8. R. Filipowsky. Electrical pulse communication systems, J. Brit. Instn. Radio Engrs., 1955, No. 9 & No. 10; 1956, No. 1.

9. C. E. Shannon. Communication in the presence of noise, Proc. IRE, 37, No. 1, 1949.

10. D. Gabor. Theory of communication, JIEE, v. III, 93, No. 26, 1946.

11. P. Hontoy, Lesuisse. Theorie de l'information caracteristiques statistiques d'un signal sinusoidal perturbe par du bruit. Revue HF, 3, No. 1, 1955.

12. A. Feinstein. A new basic theorem of information theory, Trans. IRE, PGIT-4, 1954.

13. A. Ya. Khinchin. The Concept of Entropy in Probability Theory. "Upekhi Matematicheskikh Nauk", v. VIII, No. 3, 1953.

14. M. S. Pinsker. Quantity of Information about a Gaussian Random Stationary Process, Contained in a Second Process Stationarily Linked With It. DAN, V. XCIX, No. 2, 1954.

15. B. F. Samoylov. Statisticheskiye svoystva televizionnogo signala i trebovoniya k propusknoy sposobnosti kanala, (The Statistical Properties of a Television Signal and the Demands upon the Carrying Capacity of a Channel). Svyazizdat, 1955.

16. R. A. Kazar'yan, B. I. Kubshinov, M. V. Navorov. Elementy obshohey teorii svyazi (Elements of General Information Theory). Gosenergoizdat, 1957.

17. I. M. Gel'faud, A. M. Yaglom. On Calculating the Quantity of Information about a Random Function, Contained in Another such Function. "Uspekhi Matematicheskikh Nauk", V. XII, No. 1, 1957.

18. K. Kupfmuller. Informationstheorie. Jahrb. electr. Fernmeldewesens. 1955.

19. P. Neidhardt. Grundlagen und Anwendung der Informations-theorie in der Elektrotechnik. Dtsch. Elektrotechnik, No. 11 and 12, 1956.

APPENDICES

The Poisson Distribution Function

The function $\frac{\lambda^m}{m!}e^{-\lambda}$

λ \ m	0	1	2	3	4	5	6	7
0	1,000000	0,000000	0,000000	0,000000	0,000000	0,000000	0,000000	0,000000
0,1	0,904837	0,090484	0,004524	0,000151	0,000004	0,000000	0,000000	0,000000
0,2	0,818731	0,163746	0,016375	0,001092	0,000055	0,000002	0,000000	0,000000
0,3	0,740818	0,222245	0,033337	0,003334	0,000250	0,000015	0,000001	0,000000
0,4	0,670320	0,268128	0,053626	0,007150	0,000715	0,000057	0,000004	0,000000
0,5	0,606531	0,303265	0,075816	0,012636	0,001580	0,000158	0,000013	0,000001
0,6	0,548812	0,329287	0,098786	0,019757	0,002964	0,000356	0,000036	0,000003
0,7	0,496585	0,347610	0,121663	0,028388	0,004968	0,000810	0,000080	0,000010
0,8	0,449329	0,359463	0 143785	0,038343	0,007669	0,001227	0,000164	0,000019
0,9	0,406570	0,365913	0,164661	0,049398	0,011115	0,002001	0,000300	0,000039
1,0	0,367879	0,367879	0,183940	0,061313	0,015328	0,003066	0,000511	0,000077
2,0	0,135335	0,270671	0,270671	0,180447	0,090224	0,036089	0,012030	0,003437
3,0	0,049787	0,149361	0,224042	0,224042	0,168031	0,100819	0,050409	0,021604
4,0	0,018316	0,073263	0,141525	0,195367	0,195367	0,156293	0,104194	0,059540
5,0	0,006738	0,033690	0,084224	0,140374	0,175467	0,175467	0,146223	0,104445

Appendix I

The Poisson Distribution Function

The function $\frac{\lambda^m}{m!}e^{-\lambda}$

λ \ m	0	1	2	3	4	5	6	7
0	1,000000	0,000000	0,000000	0,000000	0,000000	0,000000	0,000000	0,000000
0,1	0,904837	0,090484	0,004524	0,000151	0,000004	0,000000	0,000000	0,000000
0,2	0,818731	0,163746	0,016375	0,001092	0,000055	0,000002	0,000000	0,000000
0,3	0,740818	0,222245	0,033337	0,003334	0,000250	0,000015	0,000001	0,000000
0,4	0,670320	0,268128	0,053626	0,007150	0,000715	0,000057	0,000004	0,000000
0,5	0,606531	0,303265	0,075816	0,012636	0,001580	0,000158	0,000013	0,000001
0,6	0,548812	0,329287	0,098786	0,019757	0,002964	0,000356	0,000036	0,000003
0,7	0,496585	0,347610	0,121663	0,028388	0,004968	0,000810	0,000080	0,000010
0,8	0,449329	0,359463	0,143785	0,038343	0,007669	0,001227	0,000164	0,000019
0,9	0,406570	0,365913	0,164661	0,049398	0,011115	0,002001	0,000300	0,000039
1,0	0,367879	0,367879	0,183940	0,061313	0,015328	0,003066	0,000511	0,000077
2,0	0,135335	0,270671	0,270671	0,180447	0,090224	0,036089	0,012030	0,003437
3,0	0,049787	0,149361	0,224042	0,224042	0,168031	0,100819	0,050409	0,021604
4,0	0,018316	0,073263	0,141525	0,195367	0,195367	0,156293	0,104194	0,059540
5,0	0,006738	0,033690	0,084224	0,140374	0,175467	0,175467	0,146223	0,104445

The function $P_n(\varkappa \leq m) = 1 - \frac{\Gamma(\lambda, m+1)}{\Gamma(m+1)}$

Continuation

λ \ m	0	1	2	3	4	5	6	7
0	1,00000	1,00000	1,00000	1,00000	1,00000	1,00000	1,00000	1,00000
0,1	0,90484	0,99532	0,99985	0,99999	0,99999	0,99999	0,99999	0,99999
0,2	0,81873	0,98248	0,99885	0,99994	0,99999	0,99999	0,99999	0,99999
0,3	0,74082	0,96306	0,99640	0,99973	0,99997	0,99999	0,99999	0,99999
0,4	0,67032	0,93045	0,99207	0,99919	0,99992	0,99999	0,99999	0,99999
0,5	0,60653	0,90980	0,98561	0,99827	0,99974	0,99999	0,99999	0,99999
0,6	0,54881	0,87810	0,97688	0,99656	0,99955	0,99996	0,99999	0,99999
0,7	0,49659	0,84419	0,96586	0,99418	0,99917	0,99996	0,99999	0,99999
0,8	0,44933	0,80879	0,95258	0,99073	0,99867	0,99981	0,99998	0,99999
0,9	0,40657	0,77248	0,93714	0,99960	0,99753	0,99965	0,99996	0,99999
1,0	0,36788	0,73576	0,91970	0,98101	0,99634	0,99941	0,99992	0,99999
2,0	0,13534	0,40601	0,67688	0,86307	0,94255	0,98325	0,99538	0,99889
3,0	0,04979	0,19915	0,42319	0,64723	0,81292	0,91934	0,96638	0,98460
4,0	0,01832	0,09158	0,23810	0,43347	0,62792	0,81548	0,88876	0,94778
5,0	0,00674	0,04043	0,12465	0,26516	0,44049	0,61320	0,76221	0,86631
7,0	0,00091	0,00720	0,02964	0,08277	0,17299	0,30071	0,44971	0,59871
9,0	0,00012	0,00123	0,00623	0,02122	0,05496	0,11569	0,20678	0,32390

Appendix II

The Normal Law of Distribution

$$F(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{t^2}{2}}\,dt,\quad \varphi(x)=F'(x)=\frac{1}{\sqrt{2\pi}}\,e^{-\frac{x^2}{2}}$$

x	$F(x)$	$\varphi(x)$	x	$F(x)$	$\varphi(x)$
0,0	0,50000	0,39894	2,0	0,97725	0,05399
0,1	0,53983	0,39695	2,1	0,98214	0,04398
0,2	0,57926	0,39104	2,2	0,98610	0,03547
0,3	0,61791	0,38139	2,3	0,98928	0,02833
0,4	0,65542	0,36827	2,4	0,99180	0,02239
0,5	0,69146	0,35207	2,5	0,99379	0,01753
0,6	0,72575	0,33322	2,6	0,99534	0,01358
0,7	0,75804	0,31225	2,7	0,99653	0,01042
0,8	0,78814	0,28969	2,8	0,99744	0,00792
0,9	0,81594	0,26609	2,9	0,99813	0,00595
1,0	0,84134	0,24197	3,0	0,99865	0,00443
1,1	0,86433	0,21785	3,1	0,99903	0,00327
1,2	0,88493	0,19419	3,2	0,99931	0,00238
1,3	0,90320	0,17137	3,3	0,99952	0,00172
1,4	0,91924	0,14973	3,4	0,99966	0,00123
1,5	0,93319	0,12952	3,5	0,99977	0,00087
1,6	0,94520	0,11092	3,6	0,99984	0,00061
1,7	0,95543	0,09405	3,7	0,99989	0,00042
1,8	0,96407	0,07895	3,8	0,99993	0,00029
1,9	0,97128	0,06562	3,9	0,99995	0,00020

Appendix III

The Student Values Distribution of t_{α}, satisfying the equality $2\int_0^{t_\alpha} S_n(t)\,dt = \alpha$

	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	0,95	0,98	0,99	0,999
1	0,158	0,325	0,510	0,727	1,000	1,376	1,963	3,078	6,314	12,706	31,821	63,657	636,619
2	142	289	445	617	0,816	1,061	1,336	1,886	2,920	4,303	6,965	9,925	31,598
3	137	277	424	584	765	0,978	1,250	1,638	2,353	3,181	4,541	5,841	12,941
4	134	271	414	569	741	941	1,190	1,533	2,132	2,776	3,747	4,604	8,610
5	132	267	408	559	727	920	1,156	1,476	2,015	2,571	3,365	4,032	6,859
6	131	265	404	553	718	906	1,134	1,440	1,943	2,447	3,143	3,707	5,959
7	130	263	402	549	711	896	1,119	1,415	1,895	2,365	2,998	3,499	5,405
8	130	262	399	546	706	889	1,108	1,397	1,860	2,306	2,896	3,355	5,041
9	129	261	398	543	703	883	1,100	1,383	1,833	2,262	2,821	3,250	4,781
10	129	260	397	542	700	879	1,093	1,372	1,812	2,228	2,764	3,169	4,587
12	128	259	395	539	695	873	1,083	1,356	1,782	2,179	2,681	3,055	4,318
14	128	258	393	537	692	868	1,076	1,345	1,761	2,145	2,624	2,977	4,140
16	128	258	392	535	690	865	1,071	1,337	1,746	2,120	2,583	2,921	4,015
18	127	257	392	534	688	862	1,067	1,330	1,734	2,101	2,552	2,878	3,922
20	127	257	391	533	687	860	1,064	1,325	1,725	2,086	2,528	2,845	3,850
25	127	256	390	531	684	856	1,058	1,316	1,708	2,060	2,485	2,787	3,725
30	127	256	389	530	683	854	1,055	1,310	1,697	2,042	2,457	2,750	3,646
60	126	254	387	527	679	848	1,046	1,296	1,671	2,000	2,390	2,660	3,460
∞	126	253	385	524	674	842	1,036	1,282	1,645	1,960	2,326	2,576	3,291

The Delta-Function

By definition the delta-function $\delta(t - t_0)$ for any real parameter t_0 is equal to zero when $t \neq t_0$ and is unlimited when $t = t_0$

$$\delta(t-t_0)=\begin{cases}0, & t\neq t_0\\ \infty, & t=t_0.\end{cases} \tag{1}$$

The integral of this function within the limits of $-\infty$ to $+\infty$ is equal to unity

$$\int_{-\infty}^{\infty}\delta(t-t_0)\,dt=1. \tag{2}$$

Strictly speaking, the delta-function is obtained as the limiting one of a single-parameter set of functions. Many examples of such sets of curves can be pointed out. One such set, as has been pointed out in Sect. 3, Ch. II, is the set of normal distribution functions with a constant average value of a and with a variable mean-square σ.

Another example is the set of functions

$$\varphi(t,\lambda)=\frac{\lambda}{\pi(\lambda^2t^2+1)},$$

from which when $\lambda\to\infty$ we obtain a delta-function. As a third example let us examine the aggregate $e(t, \tau)$ of square pulses of unit area, whose duration is τ, and whose height is $1/\tau$

$$e(t,\tau)=\begin{cases}\frac{1}{\tau}, & t_0<t<t_0+\tau,\\ 0, & t<t_0,\ t>t_0+\tau.\end{cases} \tag{3}$$

If the duration of the pulse is caused to approach zero, then as a result of such a limit transition we obtain the delta-function

$$\delta(t-t_0)=\lim_{\tau\to 0}e(t,\tau). \tag{4}$$

The convolute of a delta-function with any limited, and continuous at the point t_0, function $f(t)$ has the following remarkable property

$$\int_{-\infty}^{\infty}f(t)\,\delta(t-t_0)\,dt=f(t_0). \tag{5}$$

The property expressed by formula (5) may be called the filtering property of a delta-function. A delta-function actually acts as a filter; multiplying an arbitrary function f(t) by $\delta(t-t_0)$ and integrating with respect to t, we isolate one value of this function $f(t_0)$, i.e., that value which corresponds to zero of the argument of the delta-function $t-t_0=0$. Let us note that in the filtering integral of (5) the integration limits of $-\infty$ and $+\infty$ may be replaced by any pair of finite numbers a and b, if only the point $t=t_0$ lies within the interval (a, b). The proof of formula (5) is obtained, if under the integral sign in place of $\delta(t-t_0)$ there is placed any function approximating it and then a limit is taken (Fig. 93).

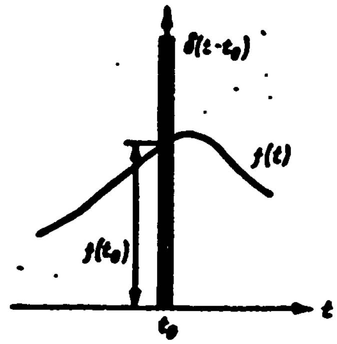


Fig. 93. Filtering property of a delta-function.

Let us find the spectrum (Fourier transformation) of a delta-function. Employing the filtering property, we obtain

$$\int_{-\infty}^{\infty} \delta(t-t_0)\, e^{-i\omega t}\, dt = e^{-i\omega t_0}. \tag{6}$$

If $t_0 = 0$, then from (6) it follows that the spectrum of $\delta(t)$ is uniform at all frequencies, with an intensity equal to unity. The spectrum of the half sum of two delta-functions $\frac{1}{2}[\delta(t+t_0)+\delta(t-t_0)]$ is, in accordance with (6), equal to $\frac{1}{2}\left(e^{i\omega t_0}+e^{-i\omega t_0}\right) = \cos\omega t_0$.

Performing now an inverse Fourier transformation, we find

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\, d\omega = \frac{1}{\pi}\int_{0}^{\infty} \cos\omega t\, d\omega = \delta(t), \tag{7}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\cos\omega_0 t\, d\omega = \frac{1}{\pi}\int_{0}^{\infty} \cos\omega t\cos\omega_0 t\, d\omega = \frac{\delta(t+t_0)+\delta(t-t_0)}{2}. \tag{7'}$$

In virtue of the symmetry of the Fourier integral, the variables t and ω in formulas

(6) and (7) may exchange places.

The derivatives of a delta-function are defined as limits of the corresponding derivatives of the approximating functions. Thus, for instance, if for such an approximation there are employed normal distribution functions when $\sigma \to 0$, then for the n-th derivative of the delta-function we obtain the following definition

$$\delta^{(n)}(t) = \lim_{\sigma \to 0} \left\{ \frac{1}{\sqrt{2\pi}\sigma} \frac{d^n}{dt^n} e^{-\frac{t^2}{2\sigma^2}} \right\}. \qquad (8)$$

Just as for the delta-function itself, its derivatives are equal to zero when $t \neq 0$. The behaviour of the derivatives when $t = 0$ is complex. Thus, for instance, the first derivative of the delta-function

$$\delta^{(1)}(t) = \lim_{\sigma \to 0} \left\{ \frac{-t}{\sqrt{2\pi}\sigma^3} e^{-\frac{t^2}{2\sigma^2}} \right\}$$

is equal to $+\infty$ when the origin of the coordinates is approached from the left ($t = 0^-$) and is equal to $-\infty$at when the origin is approached from the right ($t = 0^+$). In the vicinity of $t = 0$ the behavior of $\delta^{(1)}(t)$ is comparable with the behavior of the function $-1/t$.

The filtering property of the delta-function extends also to its derivatives. The convolute of the n-th order derivative of a delta-function with any function which has a continuous n-th-order derivative at the point t_0, is equal to

$$\int_{-\infty}^{\infty} f(t)\,\delta^{(n)}(t - t_0)\,dt = f^{(n)}(t_0). \qquad (9)$$

If the derivative $f^{(n)}(t)$ undergoes an interruption at the point t_0, then

$$\int_{-\infty}^{\infty} f(t)\,\delta^{(n)}(t - t_0)\,dt = \frac{f^{(n)}(t_0 + 0) + f^{(n)}(t_0 - 0)}{2}. \qquad (9')$$

Let us find the spectrum (Fourier transformation) of the derivative of a delta-function. Employing (9), we obtain

$$\int_{-\infty}^{\infty} \delta^{(n)}(t - t_0)\,e^{-i\omega t}\,dt = \left(\frac{d^n e^{-i\omega t}}{dt^n}\right)_{t=t_0} = (-i\omega)^n e^{-i\omega t_0}. \qquad (10)$$

If $t_0 = 0$, then from (10) it follows that the spectrum of $\delta^{(n)}(t)$ is equal to $(-i\omega)^n$.

Computation of the Integral K

$$K=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\varphi(x,y)}dxdy, \qquad (1)$$

where

$$\varphi(x,y)=a_{11}x^2+2a_{12}xy+a_{22}y^2+2a_{13}x+2a_{23}y+a_{33} \qquad (2)$$

is the positively determined quadratic form.

Expression (2) represents the equation of a second-order surface - an ellipsoid. By rotation of the axes and translation of the origin of the coordinates it is possible to reduce this equation to the canonical form

$$\Phi(u,v)=\lambda_1 u^2+\lambda_2 v^2+C, \qquad (3)$$

which, besides the free term, contains only the squares of the variables u and v.

With such a substitution of the variables, integral (1) is reduced to the form

$$K=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\Phi(u,v)}D\ dudv,$$

where D is a transformation jacobian. Since the transformation of the coordinates amounts only to a translation of the origin and a rotation of the axes, D = 1. Therefore

$$K=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(\lambda_1 u^2+\lambda_2 v^2+C)}dudv=$$

$$=e^{-\frac{C}{2}}\int_{-\infty}^{\infty} e^{-\frac{\lambda_1 u^2}{2}}du\int_{-\infty}^{\infty} e^{-\frac{\lambda_2 v^2}{2}}dv.$$

Since

$$\int_{-\infty}^{\infty} e^{-\frac{\lambda z^2}{2}}dz=\sqrt{\frac{2\pi}{\lambda}},$$

Then

$$K=\frac{2\pi}{\sqrt{\lambda_1\lambda_2}}e^{-\frac{C}{2}}. \qquad (4)$$

Thus, the computation of integral (1) has been reduced to finding the quantities

λ_1, λ_2, and C. In the theory of quadratic forms, it is proved that λ_1 and λ_2 are roots of the characteristic equation

$$\lambda^2 - I_1\lambda + I_2 = 0, \qquad (5)$$

the coefficients of which are expressed in terms of the coefficients of the quadratic form (2)

$$I_1 = a_{11} + a_{22}, \; I_2 = a_{11}a_{22} - a_{12}^2, \qquad (6)$$

and the constant C is found on the basis of the form

$$C = \frac{I_3}{I_2}, \qquad (7)$$

where

$$I_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}. \qquad (8)$$

Into formula (4) there enters only the product $\lambda_1 \lambda_2$ of the roots of equation (5), equal to the free term I_2. Therefore the desired integral is equal to

$$K = \frac{2\pi}{\sqrt{I_2}} e^{-\frac{I_3}{2I_2}}. \qquad (9)$$

Let us employ formula (9) for computing $\Theta_2(v_1, v_2)$ in #9, Chapter III. Effecting first a substitution of the integration variables $x = \frac{x_1 - a_1}{\sigma_1}$ and $y = \frac{x_2 - a_2}{\sigma_2}$, we reduce $\Theta_2(v_1, v_2)$ to the form

$$\Theta_2(v_1, v_2) = \frac{1}{2\pi\sqrt{1-r^2}} e^{i(a_1 v_1 + a_2 v_2)}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \times e^{-\frac{1}{2(1-r^2)}(x^2 - 2rxy + y^2) + iv_1\sigma_1 x + iv_2\sigma_2 y} dx dy. \qquad (10)$$

The integral in (10) is a special case of (1) when

$$a_{11} = a_{22} = \frac{1}{1-r^2}, \; a_{12} = -\frac{r}{1-r^2},$$
$$a_{13} = i\sigma_1 v_1, \; a_{23} = i\sigma_2 v_2, \; a_{33} = 0.$$

From (6) and (8) we find

$$I_2 = \frac{1}{(1-r^2)^2} - \frac{r^2}{(1-r^2)^2} = \frac{1}{1-r^2},$$

$$I_3=\begin{vmatrix} \frac{1}{1-r^2} & -\frac{r}{1-r^2} & i\sigma_1 v_1 \\ -\frac{r}{1-r^2} & \frac{1}{1-r^2} & i\sigma_2 v_2 \\ i\sigma_1 v & i\sigma_2 v_2 & 0 \end{vmatrix}=\frac{2r\sigma_1\sigma_2 v_1 v_2+\sigma_1^2 v_1^2+\sigma_2^2 v_2^2}{1-r^2}.$$

Then

$$\Theta_2(v_1, v_2)=\frac{1}{2\pi\sqrt{1-r^2}}e^{i(a_1 v_1+a_2 v_2)}2\pi\sqrt{1-r^2}\times$$

$$\times e^{-\frac{1}{2}(\sigma_1^2 v_1^2+2r\sigma_1\sigma_2 v_1 v_2+\sigma_2^2 v_2^2)},$$

which does not differ in substance from (3.95).

By means of formula (9) there is also computed the integral (8.23), which is a special case of the integral (1) when

$$a_{11}=\frac{1}{\sigma^2(1-R_0^2)}-2iv_1,\quad a_{22}=\frac{1}{\sigma^2(1-R_0^2)}-2iv_2,$$

$$a_{12}=\frac{R_0}{\sigma^2(1-R_0^2)},\qquad a_{13}=a_{23}=a_{33}=0.$$

From (6) and (8) we find

$$I_2=\frac{1}{\sigma^4(1-R_0^2)}-\frac{2i(v_1+v_2)}{\sigma^2(1-R_0^2)}-4v_1v_2,\quad I_3=0.$$

Then, employing (9), we find

$$\frac{1}{2\pi\sigma^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{iv_1x_1^2+iv_2x_2^2}e^{-\frac{x_1^2+x_2^2-2R_0x_1x_2}{2\sigma^2(1-R_0^2)}}dx_1dx_2=$$

$$=\frac{1}{2\pi\sigma^2\sqrt{1-R_0^2}}\frac{2\pi}{\sqrt{\frac{1}{\sigma^4(1-R_0^2)}-\frac{2i(v_1+v_2)}{\sigma^2(1-R_0^2)}-4v_1v_2}}=$$

$$=\frac{1}{\sqrt{1-2i\sigma^2(v_1+v_2)-4\sigma^4(1-R_0^2)v_1v_2}}.$$

Appendix VI

The Hypergeometric Function

The general expression of the hypergeometric function is provided by the following series

$$_rF_s(a_1,\dots a_r,\gamma_1,\dots\gamma_s;\ x)=$$

$$=\frac{\Gamma(\gamma_1)\Gamma(\gamma_2)\dots\Gamma(\gamma_s)}{\Gamma(a_1)\Gamma(a_2)\dots\Gamma(a_r)}\sum_{n=0}^{\infty}\frac{\Gamma(a_1+n)\Gamma(a_2+n)\dots\Gamma(a_r+n)}{\Gamma(\gamma_1+n)\Gamma(\gamma_2+n)\dots\Gamma(\gamma_s+n)}\cdot\frac{x^n}{n!}. \qquad (1)$$

When r = 2, s = 1, there is obtained the conventional hypergeometric series

$$
{}_2F_1(\alpha, \beta, \gamma, x) = 1 + \frac{\alpha\beta}{\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + \\ + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}\cdot\frac{x^3}{3!} + \dots \quad (2)
$$

When $\alpha = \beta = \gamma = 1$, series (2) is transformed into a geometric progression with the denominator x.

Another important special case of function (1), extensively employed in this book, is the degenerate (or confluent) hypergeometric function (r = 1, s = 1)

$$
{}_1F_1(\alpha, \gamma, x) = 1 + \frac{\alpha}{\gamma}x + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{x^3}{3!} + \dots \quad (3)
$$

The value of this function when $x > 0$ is linked with its value for $-x$ by the relationship

$$
{}_1F_1(\alpha, \gamma, x) = e^x {}_1F_1(\gamma-\alpha, \gamma, -x). \quad (4)
$$

For large negative values of the argument x, there takes place the asymptotic expansion

$$
{}_1F_1(\alpha, \gamma, -x) \sim \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}\cdot\frac{1}{x^\alpha}\left[1 + \frac{\alpha(\alpha-\gamma+1)}{x} + \right. \\ \left. + \frac{\alpha(\alpha+1)(\alpha-\gamma+1)(\alpha-\gamma+2)}{2x^2} + \dots\right]. \quad (5)
$$

If $\alpha = -n$ (n is a positive integer), then ${}_1F_1(-n, \gamma, x)$ is transformed into a polynomial of the n-th power with respect to x. When $\alpha = n$, $\gamma = m$ (n and m are integers), ${}_1F_1(n, m, x)$ is expressed in terms of polynomials and exponential functions of the argument x. If $\alpha = \frac{n}{2}$ and $\gamma = m$, then ${}_1F_1(n/2, m, x)$ is expressed by means of exponential and Bessel functions of the argument x. Thus, for instance,

$$
{}_1F_1\left(\frac{1}{2}, 1, -x\right) = e^{-\frac{x}{2}} I_0\left(\frac{x}{2}\right), \quad (6)
$$

$$
{}_1F_1\left(-\frac{1}{2}, 1, -x\right) = e^{-\frac{x}{2}}\left[(1+x)I_0\left(\frac{x}{2}\right) + xI_1\left(\frac{x}{2}\right)\right], \quad (6')
$$

$$
{}_1F_1\left(\frac{1}{2}, 2, -x\right) = e^{-\frac{x}{2}}\left[I_0\left(\frac{x}{2}\right) + I_1\left(\frac{x}{2}\right)\right]. \quad (6'')
$$

There is expressed in terms of a degenerate hypergeometric function the following frequently encountered integral of the product of a potential, a Bessel and an

exponential function

$$\int_0^\infty t^{\mu-1} J_\nu(\alpha t) e^{-\beta^2 t^2} dt = \frac{\Gamma\left(\frac{\mu+\nu}{2}\right)\left(\frac{\alpha}{2\beta}\right)^\nu}{2\beta^\mu \Gamma(\nu+1)} {}_1F_1\left(\frac{\mu+\nu}{2}, \nu+1, -\frac{\alpha^2}{4\beta^2}\right). \quad (7)$$

By a substitution of α for $i\alpha$ with account taken of (4), from (7) there is also obtained the expression of the integral containin_ the Bessel function of the imaginary argument

$$\int_0^\infty t^{\mu-1} I_\nu(\alpha t) e^{-\beta^2 t^2} dt = \frac{\Gamma\left(\frac{\mu+\nu}{2}\right)\left(\frac{\alpha}{2\beta}\right)^\nu}{2\beta^\mu \Gamma(\nu+1)} e^{\frac{\alpha^2}{4\beta^2}} {}_1F_1\left(\frac{\nu-\mu}{2}+1, \nu+1, -\frac{\alpha^2}{4\beta^2}\right). \quad (7')$$

In terms of degenerated hypergeometric functions it is also possible to express the derivatives of the function $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$$\varphi^{(n)}(x) = 2^{\frac{n-1}{2}} \left\{ \frac{{}_1F_1\left(\frac{1+n}{2}, \frac{1}{2}, -\frac{x^2}{2}\right)}{\Gamma\left(\frac{1-n}{2}\right)} + x\sqrt{2} \frac{{}_1F_1\left(\frac{2+n}{2}, \frac{3}{2}, -\frac{x^2}{2}\right)}{\Gamma\left(-\frac{n}{2}\right)} \right\}. \quad (8)$$

Appendix VII

Hermite Polynomials

The Hermite polynomials $H_n(x)$ are defined by the relationship

$$H_n(x) = (-1)^n e^{-\frac{x^2}{2}} \frac{d^n}{dx^n}\left(e^{-\frac{x^2}{2}}\right), \; n = 0, 1, 2, \ldots \quad (1)$$

By reiterated partial integration it is not difficult to show that

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-\frac{x^2}{2}} dx = \begin{cases} n!, & m = n, \\ 0, & m \neq n, \end{cases} \quad (2)$$

wherefrom it can be seen that Hermite polynomials represent an aggregate of orthogonal polynomials.

From the definition (1), it follows that $H_n(x)$ is a polynomial of the n-th power, this polynomial containing only even powers of x when n is even and only odd powers of x when n is odd.

Any three consecutive Hermite polynomials are linked by the recurrent relationship

$$H_{n+1}(x)=xH_n(x)-nH_{n-1}(x). \tag{3}$$

The first five Hermite Polynomials have the form

$$H_0(x)=1,\ H_1(x)=x,\ H_2(x)=x^2-1,$$
$$H_3(x)=x^3-3x,\ H_4(x)=x^4-6x^2+3.$$

The expressions for Hermite polynomials of a higher order are obtained by means of (3).

Expanding the function : $f(t)=e^{-\frac{t^2}{2}+tx}$ into a Taylor series, we obtain

$$e^{-\frac{t^2}{2}+tx}=\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n. \tag{4}$$

From (4) when x = 0, employing the well-known expansion into a series of the function $e^{-\frac{t^2}{2}}$ and comparing the coefficients for equal powers of t in the left-hand and right-hand parts, we find

$$H_{2n}(0)=(-1)^n(2n-1)!!,\ H_{2n-1}(0)=0. \tag{5}$$

Appendix VIII

Inverse Fourier Transformation of $\Theta_2(v_1, v_2, \tau)$.

Let us find the inverse Fourier transformation of the two-dimensional characteristic function (3.24), i.e., let us calculate the double integral

$$W_2(\rho_1,\rho_2,\tau)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-i(\rho_1v_1+\rho_2v_2)}dv_1dv_2}{1-2i\sigma^2(v_1+v_2)-4\sigma^4v_1v_2\left(1-R_0^2\right)}. \tag{1}$$

Let us represent the integrand function in the form of a product of two factors, one of which depends only on one integration variable.

$$\frac{e^{-i(\rho_1v_1+\rho_2v_2)}}{1-2i\sigma^2(v_1+v_2)-4\sigma^4v_1v_2\left(1-R_0^2\right)}=$$
$$=\frac{e^{-i\rho_2v_2}}{1-2i\sigma^2v_2\left(1-R_0^2\right)}\cdot\frac{e^{-i\rho_1v_1}}{\dfrac{1-2i\sigma^2v_2}{1-2i\sigma^2v_2\left(1-R_0^2\right)}-2i\sigma^2v_1}. \tag{2}$$

Employing (2), we first integrate with respect to the variable v_1, considering the other variable v_2 to be fixed, i.e., we calculate the integral

$$K(v_2)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{e^{-i\rho_1 v_1}dv_1}{\frac{1-2i\sigma^2 v_2}{1-2i\sigma^2 v_2\left(1-R_0^2\right)}-2i\sigma^2 v_1}. \tag{3}$$

By a substitution of the integration variable

$$\frac{1-2i\sigma^2 v_2}{1-2i\sigma^2 v_2\left(1-R_0^2\right)}-2i\sigma^2 v_1=2\sigma^2 z \tag{4}$$

the integral (3) is reduced to the form

$$K(v_2)=\frac{1}{2\sigma^2}e^{-\frac{\rho_1}{2\sigma^2}\cdot\frac{1-2i\sigma^2 v_2}{1-2i\sigma^2 v_2\left(1-R_0^2\right)}}\cdot\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\frac{e^{\rho_1 z}}{z}dz. \tag{5}$$

In the theory of gamma-functions it is proved* that

$$\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\frac{e^{\rho_1 z}}{z}dz=\begin{cases}1, & \rho_1>0,\\ 0, & \rho_1<0.\end{cases} \tag{6}$$

Substituting (6) into (5), we find

$$\begin{aligned}K(v_2)&=\frac{1}{2\sigma^2}e^{-\frac{\rho_1}{2\sigma^2}\cdot\frac{1-2i\sigma^2 v_2}{1-2i\sigma^2 v_2\left(1-R_0^2\right)}}, && \rho_1>0,\\ K(v_2)&=0, && \rho_1<0.\end{aligned} \tag{7}$$

Employing (7), we find

$$\begin{aligned}W_2(\rho_1,\rho_2,\tau)&=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{e^{-i\rho_2 v_2}}{1-2i\sigma^2 v_2\left(1-R_0^2\right)}K(v_2)\,dv_2=\\ &=\frac{1}{2\sigma^2}\cdot\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{1}{1-2i\sigma^2 v_2\left(1-R_0^2\right)}e^{-i\rho_2 v_2-\frac{\rho_1}{2\sigma_2}\cdot\frac{1-2i\sigma^2 v_2}{1-2i\sigma^2 v_2\left(1-R_0^2\right)}}dv_2.\end{aligned} \tag{8}$$

After simple transformations and after a substitution of the integration variable

$$1-2i\sigma^2 v_2\left(1-R_0^2\right)=R_0\sqrt{\frac{\rho_1}{\rho_2}}\,z \tag{9}$$

integral (8) is reduced to the form

$$W_2(\rho_1,\rho_2,\tau)=\frac{e^{-\frac{\rho_1+\rho_2}{2\sigma^2\left(1-R_0^2\right)}}}{4\sigma^4\left(1-R_0^2\right)}\cdot\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}e^{\frac{R_0\sqrt{\rho_1\rho_2}}{2\sigma^2\left(1-R_0^2\right)}\left(z+\frac{1}{z}\right)}\frac{dz}{z},\qquad \rho_1>0. \tag{10}$$

The integral in the right-hand part of (10) coincides with the well-known

---

* Cf., e.g., B. Van der Pohl and H. Brenner. Operatsionnoye ischesleniye na osnove dvukhstoronnego preobrazovaniya Laplasa (Operational Calculus on the Basis of a Reversible Laplace Function). For. Lit. Pub. House, 1952, p. 38.

integral representation of a zero-order Bessel function of an imaginary argument (Cf. G. N. Watson, Teoriya besselevykh funktsiy (Theory of Bessel Functions). For. Lit. Pub. Hse, 1949, p. 200).

Thus the desired two-dimensional distribution function is equal to

$$W_2(\rho_1, \rho_2, \tau) = \frac{1}{4\sigma^4(1-R_0^2)} e^{-\frac{\rho_1+\rho_2}{2\sigma^2(1-R_0^2)}} I_0\left[\frac{R_0\sqrt{\rho_1\rho_2}}{\sigma^2(1-R_0^2)}\right],$$

$$\rho_1>0,\ \rho_2>0,$$

$$W_2(\rho_1, \rho_2, \tau)=0,\ \rho_1<0,\ \rho_2<0,$$

which does not differ from (8.19).

Appendix IX

Nicholson's Function

Let us investigate the integral

$$K(\theta) = \frac{2s}{\sqrt{2\pi}}\int_0^\theta \cos\varphi F(s\cos\varphi)\, e^{-\frac{s^2\sin^2\varphi}{2}} d\varphi, \tag{1}$$

which figures in the right-hand part of (8.57). Substituting into (1) the expression for the Laplace function F(x), we obtain

$$K(\theta) = \frac{2s}{\sqrt{2\pi}}\int_0^\theta \cos\varphi \cdot e^{-\frac{s^2\sin^2\varphi}{2}} \int_{-\infty}^{s\cos\varphi} e^{-\frac{y_1^2}{2}} dy d\varphi.$$

Effecting now the substitution of integration variables

$$x = \sin\varphi, \quad y = sy_1,$$

we find

$$K(\theta) = \frac{s^2}{\pi}\int_0^{\sin\theta}\int_{-\infty}^{\sqrt{1-x^2}} e^{-\frac{s^2}{2}(x^2+y^2)} dydx. \tag{2}$$

Let us now separately examine the two cases of $\theta \leqslant \frac{\pi}{2}$ and $\theta > \frac{\pi}{2}$.

1) $\theta \leqslant \frac{\pi}{2}$. We shall in this case designate integral (1) by $K_1(\theta)$. The area of integration for $K_1(\theta)$ is shown in Figure 94.

Let us breakdown the interval of integration along y in (2) into two sectors

$$K_1(\theta)=\frac{s^2}{\pi}\int\limits_0^{\sin\theta}\int\limits_{-\infty}^{0}e^{-\frac{s^2}{2}(x^2+y^2)}dydx+ \tag{3}$$

$$+\frac{s^2}{\pi}\int\limits_0^{\sin\theta}\int\limits_0^{\sqrt{1-x^2}}e^{-\frac{s^2}{2}(x^2+y^2)}dydx.$$

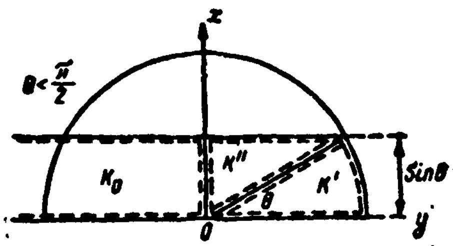


Fig. 94. Areas of integration in the integral $K_1(\theta)$.

In the first integral the variables separate

$$K_0=\frac{s^2}{\pi}\int\limits_0^{\sin\theta}\int\limits_{-\infty}^{0}e^{-\frac{s^2}{2}(x^2+y^2)}dydx=$$

$$=\frac{2}{\sqrt{2\pi}}\int\limits_0^{\sin\theta}e^{-\frac{s^2x^2}{2}}d(sx)\cdot\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{0}e^{-\frac{s^2y^2}{2}}d(sy)= \tag{4}$$

$$=2\left[F(s\sin\theta)-\frac{1}{2}\right]\cdot\frac{1}{2}=F(s\sin\theta)-\frac{1}{2}.$$

The second integral is broken down into two: K' and K'' (cf. Fig. 94). Integral K' is easily calculated by a transformation to polar coordinates

$$K'=\frac{s^2}{\pi}\int\limits_0^{\theta}\int\limits_0^{1}\rho e^{-\frac{s^2\rho^2}{2}}d\rho d\varphi=\frac{\theta}{\pi}\left(1-e^{-\frac{s^2}{2}}\right). \tag{5}$$

Integral K'' along the triangular area may be written down in the form

$$K''=\frac{s^2}{\pi}\int\limits_0^{\sin\theta}\int\limits_0^{x\operatorname{ctg}\theta}e^{-\frac{s^2}{2}(x^2+y^2)}dydx= \tag{6}$$

$$=\frac{1}{\pi}\int\limits_0^{s\sin\theta}\int\limits_0^{x_1\operatorname{ctg}\theta}e^{-\frac{1}{2}(x_1^2+y_1^2)}dy_1dx_1.$$

The integral

$$V(h,q)=\frac{1}{2\pi}\int\limits_0^{h}\int\limits_0^{\frac{qx}{h}}e^{-\frac{x^2+y^2}{2}}dydx \tag{7}$$

has been studied by Nicholson (cf. Biometrica, V. 38, 1943). The cited work also presents tables of this integral for a change in the parameters h and q from o to 3 at intervals of 0.1. Comparing (6) and (7), we find

$$K'' = 2V(s\sin\theta;\ s\cos\theta). \tag{8}$$

Substituting (4), (5) and (8) into (2), we obtain

$$K_1(\theta) = F(s\sin\theta) - \frac{1}{2} + \frac{\theta}{\pi}\left(1 - e^{-\frac{s^2}{2}}\right) + \\ + 2V(s\sin\theta,\ s\cos\theta). \tag{9}$$

From (7) it can be seen that $V(0,q) = V(h,0) = 0$. Therefore $K(0)=0$ and

$$K_1\left(\frac{\pi}{2}\right) = F(s) - \frac{1}{2}e^{-\frac{s^2}{2}}.$$

2) $\theta \geqslant \frac{\pi}{2}$. We shall in this case designate integral (1) by $K_2(\theta)$. Let us break down the interval of integration along θ into two:

$$K_2(\theta) = \frac{2s}{\sqrt{2\pi}}\int_0^{\frac{\pi}{2}} \cos\varphi F(s\cos\varphi)e^{-\frac{s^2\sin^2\varphi}{2}}d\varphi + \\ + \frac{2s}{\sqrt{2\pi}}\int_{\frac{\pi}{2}}^{\theta} \cos\varphi F(s\cos\varphi)e^{-\frac{s^2\sin^2\varphi}{2}}d\varphi. \tag{10}$$

The first integral coincides with $K_1(\frac{\pi}{2})$. In the second integral we effect the substitution $\psi = \pi - \varphi$.

Then

$$K_{21}(\theta) = \frac{2s}{\sqrt{2\pi}}\int_{\frac{\pi}{2}}^{\theta} \cos\varphi F(s\cos\varphi)e^{-\frac{s^2\sin^2\varphi}{2}}d\varphi =$$

$$= -\frac{2s}{\sqrt{2\pi}}\int_{\pi-\theta}^{\frac{\pi}{2}} \cos\psi F(-s\cos\psi)e^{-\frac{s^2\sin^2\psi}{2}}d\psi,$$

and since $F(-s\cos\psi) = 1 - F(s\cos\psi)$, then

$$K_{21}(\theta) = -\frac{2s}{\sqrt{2\pi}}\int_{\pi-\theta}^{\frac{\pi}{2}} \cos\psi[1 - F(s\cos\psi)]e^{-\frac{s^2\sin^2\psi}{2}}d\psi =$$

$$= -\frac{2}{\sqrt{2\pi}} \int_{\pi-\theta}^{\frac{\pi}{2}} d(s\sin\psi)\, e^{-\frac{s^2\sin^2\psi}{2}} +$$

$$+ \frac{2s}{\sqrt{2\pi}} \int_{\pi-\theta}^{\frac{\pi}{2}} \cos\psi\, F(s\cos\psi)\, e^{-\frac{s^2\sin^2\psi}{2}} d\psi =$$

$$= 2[F(s\sin\theta) - F(s)] + K_1\left(\frac{\pi}{2}\right) -$$

$$- \frac{2s}{\sqrt{2\pi}} \int_{0}^{\pi-\theta} \cos\psi F(s\cos\psi)\, e^{-\frac{s^2\sin^2\psi}{2}} d\psi,$$

or, taking into account (7), we find

$$K_{21}(\theta) = 2[F(s\sin\theta) - F(s)] + K_1\left(\frac{\pi}{2}\right) - K_1(\pi - \theta). \qquad (11)$$

Substituting (11) into (10) and taking into account (9), we obtain

$$K_2(\theta) = 2K_1\left(\frac{\pi}{2}\right) + 2F(s\sin\theta) - 2F(s) - F(s\sin\theta) +$$

$$+ \frac{1}{2} - \frac{\pi-\theta}{\pi}\left(1 - e^{-\frac{s^2}{2}}\right) - 2V(s\sin\theta, -s\cos\theta),$$

or

$$K_2(\theta) = F(s\sin\theta) - \frac{1}{2} +$$

$$+ \frac{\theta}{\pi}\left(1 - e^{-\frac{s^2}{2}}\right) - 2V(s\sin\theta, -s\cos\theta). \qquad (12)$$

List of the Most Generally Used Notations

(Numbers indicate the pages on which the designations were introduced)

A - random event, 2.

$\bar{A}$ - random event, converse to A, 4.

$A(t)$ - random function, 237.

A_0 - amplitude of carrier oscillation, 413.

a - parameter of normal distribution, 46.

B - random event, 6.

$B(\tau)$ - correlation function, 168.

$B_0(\tau)$ - correlation function of envelope, 307.

$B_{A}(\tau)$ - correlation function corresponding to the discrete part of the power spectrum, 289.

$B_{H}(\tau)$ - correlation function corresponding to the continuous part of the power spectrum, 394.

$B^*(\tau)$ - correlation function, averaged over time, of a nonstationary random process, 257.

C - carrying capacity of a channel, 436.

$C(t)$ - random function, 237.

$C(\omega)$ - frequency characteristic of a linear system, 216.

C_n^k - binomial coefficient, 16.

D - determined, 57, transformation jacobian, 89.

D_{ik} - algebraic supplement (Algebraicheskoye dopolneniye), 57.

E - envelope of a set of curves, 236.

E - elliptical integral of the 2-d kind, 264.

e - base of natural logarithms (sometimes the symbol <u>exp</u> is used).

F - Laplace function, 30, one-dimensional integral distribution function, 40, highest frequency in a spectrum, 439.

F_n - n-dimensional integral distribution function, 55.

F_{1k} - one-dimensional integral distribution function of the random variable ξ_k, 55.

${}_1F_1$ - degenerate hypergeometric function, 457.

F_0 - intensity of white noise, 204.

$F(\omega)$ - power spectrum, 188.

$f(x)$ - characteristic of a nonlinear system, 216.

G - region of space, 55.

$g(\omega)$ - pulse spectrum, 382.

H - entropy, 80.

H' - rate of production of information, 429.

H_n - Hermite polynomial of the n-th order, 458.

$H(t)$ - Fourier transformation of the square of a frequency characteristic, 220.

$H(\omega)$ - spectrum component of a pulse process, 380.

$h(t)$ - pulse transition function, 219.

h_{nk} - coefficients of the expansion into a series of a correlation function, 283.

I_n - n-th order Bessel function of an imaginary argument, 94.

I - quantity of information, 433.

I' - rate of transmission of information, 435.

K - kernel of an integral transformation, 355.

K - full elliptical integral of the first kind, 264.

$K(\omega)$ - spectrum component of a pulse process, 379.

k - [italicized] - event occurrence number with independent tests, 15.

k - coefficient of asymmetry, 69.

$k(i\omega)$ - transmission function of a linear system, 217.

L_n^{α} - Laguerre polynomial, 306.

M - mean number of symbols emitted in a unit of time, 429.

M_k - central moment of the k-th order, 64.

m_k - initial moment of the k-th order, 63.

M_2 - dispersion, 65.

m_1 - mean value, 64.

m - index of modulation, 422.

P, p - probabilities, 3.

P_c - power of signal, 183.

$P_{\text{ш}}$ - power of noise, 183.

Q_n - orthogonal polynomials, 241.

q - probability, 15.

R - correlation coefficient, 73, 181.

R_u - coefficient of excess, 431.

S_n - Student distribution, 1[illegible].

S(t) - determined part of a random process, 165.

s - ratio of signal amplitude to the mean-square value of noise, 321.

T - time of observation, period of repetition, [no page reference given].

T_n - Chebyshev polynomial of the n-th order, 334

t - current time, (Standardized measurement error, 153). [sic.]

u_o - potential difference, 2

u(t) - envelope of determined signal, 270.

V - Nicholson function, 461.

W, W, W - distribution functions (probabilities densities), 42.

W_n - n-dimensional distribution function, 56.

W_{1k} - one-dimensional distribution function of the random variable ξ_k, 57.

x, y, z - arguments of distribution functions, 45.

Z_{KT} - spectrum of piece of realization, 185.

α - parameter of the generalized Rayleigh function 100, reliability 152, circuit damping, 229.

α_1, α_2 - non-dimensional magnitudes of determined signals, 254, 305.

β - magnitude, proportional to the transmission band, 228.

Γ - gamma-function, 31.

γ - coefficient of excess, 70.

δ - delta-function, 451.

Δf - width of transmission band, 224.

ε - accuracy, 151.

θ - characteristic function, 107.

θ_n - nondimensional characteristic function, 111.

λ - Poisson-distribution parameter, 29.

μ - power ratio of continuous and discrete spectra, 382 coefficient of compression, 432.

ν - relative occurrence frequency of an event, 2 , degree of nonlinearity, 284.

σ - parameter of normal law of distribution, 46.

Φ - Kramp function, 26.

φ, Φ, ψ - phase [no page reference given]

Ω - modulation frequency, 273.

ω - present frequency [no page reference given]

ω_0 - central frequency of spectrum, 202 , frequency of carrier oscillation, 413.

$\omega_1^2 =$ - $R''(0)$, 210.

τ - temporal shift [no page reference given]

τ_0 - correlation time, 181 , pulse duration, 394.

$\xi, \eta,$ - random variables, 45.

$\xi(t), \eta(t)$ - random functions, 159.

$\xi^{\nu}(t)$ - realization of a random function, 160.

$\binom{n}{k}$ - binomial coefficient, 16.

$(2k-1)!!$ - product of all odd numbers of a natural series to 2k - 1, inclusive, 114

$(2k)!!$ - product of all odd numbers of a natural series to 2k, inclusive, 115.

$O(x)$ - order of magnitude of x, 23.

SUBJECT INDEX

A

B

C

E

P

Q

R